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A Pushing Up Result and Some Consequences for the Embedding of 2-Constrained Subgroups

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TO DANY GORENSTEIN ON HIS 65TH BIRTHDAY

1. INTRODUCTION

Let M_0, M_1 be maximal 2-local subgroups of some finite group G with $O_2(G)=1$ satisfying $Q_i=O_2(M_i)=F^*(M_i)$ for $i=0, 1$. Is it possible in this situation that $Q_1 \geqslant Q_0$? The bulk of the proof of [1] consists of showing that there exists no K-group G of characteristic 2-type satisfying these and certain non-generational conditions for M_1, which by results of G. Seitz strongly restrict the structure of M_1. If one tries to consider this problem more generally, one realizes that it splits naturally into two cases:

(1) $Z_1=\Omega_1(Z(Q_1)) \leqslant Z(O^{2'}(M_0 \cap M_1))$,

(2) Z_1 is an FF-module for $O^{2'}(M_0 \cap M_1)$.

(In both cases $Z_0=\langle Z_1^{M_0}\rangle$ is an FF-module for $O^{2'}(M_0)$!) While it is probably still not possible to treat case (2) in generality, without an extra condition assuring that one gets additional action on Z_1 (i.e., $M_1 \neq C_{M_1}(Z_1)(M_0 \cap M_1)$!), the treatment of case (1) is possible and easy, thanks to the amalgam-method, which allows for better pushing up theorems.

In this paper we prove:

THEOREM 1. *Suppose G is a finite group with $Q_0=O_2(G)=F^*(G)$ and $F^*(\bar{G})$ is a known quasisimple group, where $\bar{G}=G/Q_0$. Let $S \in \mathrm{Syl}_2(G)$, $1 \neq Z \leqslant \Omega_1(Z(S))$, and $Q=O_2(N_G(Z))$ and suppose there exists some $Q_0 \leqslant Q_1 \leqslant Q$, with $Q_1 \lhd N_G(Z)$ and:*

(∗) *There exists no characteristic subgroup of Q_1 which is normal in G.*

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Set $\bar{G}_0 = F^*(\bar{G})\tilde{Q}_1$, $R_0 = O^2(G_0)$, *and* $N_0 = Q_0 \cap R_0$. *Then one of the following holds*:

(a) $\bar{G}_0 \simeq \mathrm{SL}_n(q)$, $n \geqslant 2$, $q = 2^m \geqslant 2$, $Q_1 = Q$, *and we have one of the following cases*:

(a1) $n \geqslant 3$, $\Phi(N_0) = 1$, *and* N_0 *is the natural module for* $R_0/N_0 \simeq \mathrm{SL}_n(q)$.

(a2) $n = 2$, $\Phi(N_0) = 1$, $N_0 = [N_0, R_0]$, *and* $N_0/C_{N_0}(R_0)$ *is the natural module for* $R_0/N_0 \simeq \mathrm{SL}_2(q)$.

(a3) $R_0 \simeq 2^4 \cdot \hat{A}_8$ *or* $4^3 \cdot \widehat{\mathrm{SL}_3(4)}$, *where* $\hat{A}_8$ *or* $\widehat{\mathrm{SL}_3(4)}$ *are perfect central extensions of* A_8 *resp.* $\mathrm{SL}_3(4)$ *by an elementary* 2-*group of order* 2 *resp.* $\leqslant 4$.

(a4) $\Phi(N_0) \simeq \mathbb{Z}_2$, $R_0/N_0 \simeq \mathrm{SL}_4(2) \simeq \Omega^+(6, 2)$, $Z(N_0)/\Phi(N_0)$ *is the natural* $\mathrm{SL}_4(2)$-*module, and* $N_0/Z(N_0)$ *is the natural* $\Omega^+(6, 2)$-*module. Moreover*, $N_0/\Phi(N_0)$ *is an indecomposable* R_0-*module.*

(b) $R_0/N_0 \simeq G_2(q)'$, $q = 2^m$, $\Phi(N_0) = 1$, $N_0 = [N_0, R_0]$, *and* $N_0/C_{N_0}(R_0)$ *is the natural* $G_2(q)$-*module.* (*Of dimension* 6 *over* $\mathrm{GF}(q)$!) *Further* $|Q_1 : Q_0| = q^3$ *if* $q > 2$.

(c) $\bar{G}_0 \simeq \mathrm{Sp}(2n, q)$, $q = 2^m$, $n \geqslant 2$, $Q_1 = Q$, *and one of the following holds*:

(c1) $R_0/N_0 \simeq \mathrm{Sp}(6, q)$, $\Phi(N_0) \leqslant Z(N_0)$ *has order* q, $Z(N_0)$ *is the natural* $O_7(q)$-*module for* R_0, *and* $N_0/Z(N_0)$ *is the irreducible* R_0-*module of order* q^8 *obtained from* $\mathrm{Sp}(6, q) \subset \Omega^+(8, q)$.

(c2) $R_0/N_0 \simeq \mathrm{Sp}(2n, 2)'$, $\Phi(N_0) = 1$, $N_0 = [N_0, R_0]$, *and* $N_0/C_{N_0}(R_0)$ *is the natural* $\mathrm{Sp}(2n, 2)'$-*module.*

(c3) $R_0/N_0 \simeq \mathrm{Sp}(2n, 4)$, $\Phi(N_0) = 1$, $N_0 = [N_0, R_0]$, *and* $N_0/C_{N_0}(R_0)$ *is the natural* $\mathrm{Sp}(2n, 4)$-*module. Further, the extension of* R_0/N_0 *by* N_0 *does not split.*

(d) $R_0/N_0 \simeq \mathrm{Sp}(2n, q)'$, $q = 2^m \geqslant 2$, $|Q_1 : Q_0| = q$, $\Phi(N_0) = 1$, $N_0 = [N_0, R_0]$, *and* $N_0/C_{N_0}(R_0)$ *is the natural* R_0/N_0-*module. Further, the extension of* R_0/N_0 *by* N_0 *splits if* $q \geqslant 8$.

A $2'$-component of some finite group G is a subnormal subgroup $R = R^{(\infty)}$ with $O_2(R) = F^*(R)$ and $R/O_2(R)$ quasisimple.

THEOREM 2. *Suppose* G *is a finite* K-*group with* $O_2(G) = F^*(G)$. *Let* $S \in \mathrm{Syl}_2(G)$, $1 \neq Z \leqslant \Omega_1(Z(S))$, $M = N_G(Z)$, *and* $O_2(G) \leqslant Q_1 \leqslant O_2(M)$ *and suppose*:

(∗) $Q_1 \lhd M$ *and* $C(G, Q_1) \leqslant M < G$.

Then one of the following holds:

(1) *There exists an* $A_4 \simeq R_0 \lhd\lhd G$, $R_0 \nleq M$.

(2) *There exists a* 2′*-component* R_0 *of* G *with* $R_0 \nleq M$ *and for each such* 2′*-component* R_0 *we have* $Q_1 \leqslant N(R_0)$ *and one of the cases* (a)–(d) *of Theorem* 1 *holds for* R_0 *and* $N_0 = O_2(R_0)$.

Theorem 2 is actually the main result of this paper, since it is not easy to reduce the hypothesis of Theorem 2 to that of Theorem 1. (See (3.4)!) Case (a4) is the most interesting case in both theorems. It is the only case in which the parameter, I call d, is equal to 4. ·2 is a completion of this case. The notation $2^4 \cdot 2^{1+6}A_8$ in the atlas is misleading, since it is easy to see that $N_0/\Phi(N_0)$ must be indecomposable.

A result similar to Theorems 1 and 2 should also hold in odd characteristic, using the same proofs. But for that purpose one needs the list of FF-modules for finite groups X with $F^*(X)$ quasisimple, which is for even characteristic a consequence of [3, 8] and which is in odd characteristic not yet in the literature.

If we now return to our original problem, not all cases of Theorem 1 can arise. (Thanks to the condition $Q_1 = F^*(M_1)$!) Precisely we have:

THEOREM 3. *Suppose* M_0, M_1 *are finite subgroups of some group* G *satisfying*:

(a) $G = \langle x, M_1 \rangle$ *for each* $x \in M_0 - M_1$.

(b) $N_i = O_2(M_i) = F^*(M_i)$, $i = 0, 1$.

(c) $N_1 \geqslant N_0$, $(N_1)_G = 1$, *and* $N_1 \leqslant M_0$.

(d) M_0 *is a* K*-group*.

(e) *There exists some* $1 \neq Z \leqslant \Omega_1(Z(T))$, $T \in \mathrm{Syl}_2(M_0)$ *with* $Z \lhd M_1$.

Suppose further that there exists a 2′*-component* K_0 *of* M_0 *with* $K_0 \nleq M_1$ *satisfying*:

(∗) $G = \langle N_{M_0}(K_0 N_1), y \rangle$ *for each* $y \in M_1 - M_0$.

Then one of the following holds:

(1) M_0 *is the non-split extension of* $\mathrm{SL}_n(2)$, $3 \leqslant n \leqslant 5$, *by its natural module*, $M_1 \in \{2^{1+4}(\mathbb{Z}_3 \times \mathbb{Z}_3)\mathbb{Z}_2,\ 2^{1+6}A_7,\ 2^{1+8}A_9\}$, *and* $M_0 \cap M_1 = 2^{1+2(n-1)}L_{n-1}(2)$.

(2) $\mathrm{SL}_n(q) \lessapprox M_0/N_0 \lessapprox \Gamma L_n(q)$, $n \geqslant 2$, $q = 2^m$; $\Phi(N_0) = 1$ *and* $\tilde{N}_0 = N_0/C_{N_0}(M_0')$ *is the natural module for* M_0/N_0. *Further* $M_0 \cap M_1 \lhd M_1$ *and is the maximal parabolic subgroup of* M_0 *stabilizing a point of* $\tilde{N}_0$. *Moreover, one of the following holds*:

(i) $n \geqslant 3$, $q > 2$. *Then* $C_{N_0}(M_0') = 1$ *and* $M_1 = (M_0 \cap M_1)\langle\sigma\rangle$, *with* $\sigma^2 \in M_0 \cap M_1$ *and* σ *inducing an outer diagram automorphism on* $M_0 \cap M_1/N_1$.

(ii) $n = 2$, $q > 2$. *Then* $|C_{N_0}(M_0')| = q$ *or* 1 *and* N_0 *is indecomposable. Further* $|M_1 : M_1 \cap M_0| = 2$.

(iii) $q = 2$, $n \geqslant 3$. *Then* $N_0 = N_0' \times C_0$; $C_0 = C_{N_0}(M_0)$, *where* $N_0' = [N_0, M_0]$ *is the natural* M_0/N_0-*module and* $|C_0| \leqslant 2$ *if* $n \geqslant 4$ *resp.* $|C_0| = 1$ *or* 4 *if* $n = 3$. *Further* $M_1 = (M_0 \cap M_1)\langle\sigma\rangle$, $\sigma^2 \in M_0 \cap M_1$, *and* σ *induces an outer diagram automorphism on* $M_0 \cap M_1/N_1$ *if* $n \geqslant 4$, *resp.* $M_1/N_1 \simeq \Sigma_3 \times \mathbb{Z}_3$ *if* $n = 3$.

(3) M_0 *is the extension of the group* R_0 *of case* (c1) *of Theorem* 1 *by field automorphisms.* $M_1 = (M_0 \cap M_1)\langle\sigma\rangle$, $\sigma^2 \in M_0 \cap M_1$, *and* σ *induces an outer diagram automorphism on* $(M_0 \cap M_1/N_1)' \simeq \mathrm{Sp}(4, q)$.

If $Z_1 = \Omega_1(Z(N_1)) \leqslant Z(T)$, $T \in \mathrm{Syl}_2(M_0 \cap M_1)$, then $T \in \mathrm{Syl}_2(M_0)$ (see (3.2)). So Theorem 3 applies to case (1) of the Introduction.

It is clear that case (1) of Theorem 3 is the only interesting case, since in all other cases $M_0 \cap M_1 \lhd M_1$ and $|M_1 : M_0 \cap M_1| \leqslant 3$. An example for $n = 3$ is $G_2(3)$ and for $n = 5$ the Thompson group. I do not know any finite example for $n = 4$, but it certainly exists as amalgamated product.

Condition (*) of Theorem 3 seems unnatural and the reader might want to replace it by a condition symmetric to (a). But in this case we would get examples of the following type: G is the extension of a direct product of $F_4(q)$'s by a group $A \times \langle\sigma\rangle$, where A is arbitrary and permutes the components while σ induces an outer diagram automorphism on each component. In this case, although it is possible to determine (roughly) the structure of M_0 (see (8.1)) I do not see a chance to determine M_1, since we determine M_1 as automorphism group of N_1. But on the other hand the following proposition will show that in many natural cases a seemingly weaker condition will suffice.

PROPOSITION 4. *Suppose G is a group satisfying* (a)–(e) *of Theorem* 3. *Suppose further that there exists no* $A_4 \simeq L_0 \lhd\lhd M_0$ *with* $L_0 \not\leqslant M_1$. *Then in the following two cases condition* (*) *of Theorem* 3 *is satisfied.*

(1) *G is finite of characteristic* 2-*type and* M_0 *is a maximal* 2-*local subgroup of G.*

(2) *There exists exactly one* $2'$-*component* K_0 *of* M_0 *with* $K_0 \not\leqslant M_1$ *and* $G = \langle M_0, y\rangle$ *for each* $y \in M_1 - M_0$.

The proof that (2) suffices for (*) is almost trivial, while the proof that (1) suffices is standard, but needs some arguments of the theory of Aschbacher blocks. It will not be given here, since to use these results for revisionism one probably needs a weakening of the characteristic 2-type condition and I do not know what the status of the block theory is under such a condition. It should be mentioned that the treatment of $\mathrm{Sp}(2n, 2^m)$ and $G_2(2^m)$ in Theorem 1 is implicitly contained in Meierfrankenfeld's thesis. But, since I did not want to quote to many things from the proofs (our hypothesis is different from Meierfrankenfeld's) and since it is very short anyway, I did not include it here.

2. Some Lemmata

(2.1) Lemma. *Suppose* $G \simeq \mathrm{Sp}(2n, 2^m)$, $m \geqslant 2$, *or* $G_2(2^m)$, $m \geqslant 1$, *and* $V = [V, G]$ *is a* $\mathrm{GF}(2)G$*-module with* $\tilde{V} = V/C_V(G)$ *the natural module. Let* $S \in \mathrm{Syl}_2(G)$ *and* $A \leqslant O_2(C_G(C_{\tilde{V}}(S))) = Q$ *an offending subgroup. Then the following holds*:

(a) $C_V(G) \leqslant [V, A]$.

(b) *If* $G \simeq \mathrm{Sp}(2n, 2^m)$ *and* $\tilde{U} \coloneqq C_{\tilde{V}}(S)^{\perp}$, *considering* $\tilde{V}$ *as symplectic space, then* $|[U, Q]| = 2^m$.

Proof. To prove (a) for $G_2(q)$ one may, by way of contradiction, assume $C_V(G) \cap [V, A] = 0$. Hence $|[V, A]| = q^3$. Since G is generated by two conjugates of A, this implies $C_V(G) = 0$ and (a) holds. For $\mathrm{Sp}(2n, q)$ (a) follows from $|[V, T]| = |C_V(G)|\, q$, when $m > 1$ and T is a root-group of transvections.

To prove (b) we may assume that V is the orthogonal $\mathrm{GF}(2)G$-module. (Obtained from $\mathrm{Sp}(2n, 2^m) \simeq \Omega(2n+1, 2^m)$.) Let $Z = C_V(S)$. Then $Z = C_V(G) \oplus P$, where P is the only singular point in Z, since $C_V(G)$ is non-singular.

Let $u \in U$ be a singular vector and $x \in Q$. Then $[u, x] \in Z$ and $[u, x] = u + u^x$ is singular, since $Z \subseteq U^{\perp}$. Hence $[u, x] \in P$. Since U is generated by singular vectors, this implies (b).

(2.2) Lemma. *Let* $G \simeq \mathrm{Sp}(2n, 2^m)$, Q *as in* (2.1), *and let* $V = \langle C_V(Q)^G \rangle$ *be a non-trivial* $\mathrm{GF}(2)G$*-module. Then there exists a submodule* U *of* V, *such that* $\tilde{V} = V/U$ *is non-trivial and irreducible.*

Proof. Let V be of minimal dimension satisfying the hypothesis, but not the conclusion of (2.2). Then $V = U + C_V(Q)$, with non-trivial irreducible submodule U. Moreover V is indecomposable.

Let $P = Q \cdot L = N_G(Q)$ and $P^- = Q^- L$ the opposite parabolic subgroup, with Levi complement L and unipotent radicals Q and Q^-. If $C_U(Q)$ is a trivial L-module, then $C_V(Q)$ is a trivial P-module and (2.2) is a consequence of Gaschütz's theorem. Hence [18] implies that $C_U(Q)$ and

$C_U(Q^-)$ are direct sums of non-trivial irreducible $\mathrm{GF}(2)L$-modules. Further $U = [U, Q] \oplus C_U(Q^-)$.

Without loss $|V:U| = 2$. By the above $V = [U, Q] \oplus C_V(Q^-)$ since $V = U + C_V(Q^-)$. Let $C = C_V(Q^-) \cap ([U, Q] + C_V(Q))$. Then $|C| = 2$ since $C_U(Q) \leqslant [U, Q]$. Since $C \leqslant C_V(Q^-)$ is L-invariant, it follows that $V = U \oplus C$, $C \leqslant C_V(P^-)$. Hence again Gaschütz's theorem implies the result.

(2.3) LEMMA. *Let $G \simeq \mathrm{SL}_4(2) \simeq \Omega^+(6, 2)$ and V a $\mathrm{GF}(2)G$-module such that the only non-trivial G-composition factor in V is the orthogonal module. Let $A < G$ be the group of all transvections corresponding to a fixed point resp. fixed hyperplane in the natural representation of G and assume $|A| \geqslant |V:C_V(A)|$. Then $V = [V, G] \oplus C_V(G)$, $[V, G]$ irreducible.*

Proof. By [10] we may assume $C_V(G) \simeq \mathbb{Z}_2$ and $V = [V, G]$, since if $C_V(G) = 0$ and $|V:[V, G]| = 2$, then V is obviously not an FF-module. Hence V is a submodule of the Σ_8 permutation module over $\mathrm{GF}(2)$, since V is uniquely determined. By [15, (2.2)] we have for an offending subgroup $A < G$,

$$|C_V(A)| = 16 = |C_V(x)| \qquad \text{for each} \quad x \in A^\#,$$

and $N_G(A) \simeq 2^3 \cdot L_3(2)$. Since V admits Σ_8 we have for some $h \in \Sigma_8 - G$: $A \cap A^h = \langle x \rangle$, a contradiction to $C_V(A) = C_V(x)$.

(2.4) LEMMA. *Let $G = \mathrm{SL}_3(q)$, $q = 2^m$, and N the natural $\mathrm{GF}(q)G$-module. Then the following hold:*

(1) *$N \otimes_{\mathrm{GF}(q)} N$ is an indecomposable $\mathrm{GF}(q)G$-module with composition factors N^*, N, N^*. (N^* the dual module.)*

(2) *$N \otimes_{\mathrm{GF}(2)} N \simeq \bigoplus_{\sigma \in \mathrm{Gal}(\mathrm{GF}(q))} (N \otimes_{\mathrm{GF}(q)} N\sigma)$ as $\mathrm{GF}(2)G$-module.*

(3) *There exists no $\mathrm{GF}(2)G$-image of $N \otimes_{\mathrm{GF}(2)} N$ which is equivalent to N, $N^* \oplus N^*$, or $N^* \oplus T$, where T is a trivial $\mathrm{GF}(2)G$-module with $|T| \geqslant 2$.*

Proof. (1) is [6, (3.2)]. Statement (2) follows with the usual field extension machinery. Statement (3) is an immediate consequence of (1) and (2), if one uses the fact that for $\sigma \neq \mathrm{id}$ by Steinberg's tensor-product theorem $N \otimes_{\mathrm{GF}(q)} N\sigma$ is a non-trivial irreducible $\mathrm{GF}(q)G$-module.

(2.5) LEMMA. (a) *Let $\bar{G} \simeq G_2(q)$, $q = 2^n > 2$, $\bar{P}$ a maximal parabolic of $\bar{G}$ which does not normalize a long root subgroup, $\bar{Q} = O_2(\bar{P})$ and $\bar{B} = Z(\bar{Q})$. Then $|\bar{B}| = q^2$, $|\bar{Q}/\bar{B}| = q^3$, and $\bar{Q}/\bar{B}$ is an indecomposable module for $(\bar{P}/\bar{Q})' \simeq L_2(q)$ which is an extension of a natural by a trivial module.*

(b) *Let G be an extension of $\bar{G}$ by an elementary abelian 2-group V satisfying $V/C_V(G)$ is the natural $\bar{G}$-module. Let P, Q, and B be the coimages of $\bar{P}$, $\bar{Q}$ resp. $\bar{B}$ in G and $S \in \mathrm{Syl}_2(P)$. Then $V = V^\alpha$ for each automorphism α of S or of Q.*

Proof. Let $\bar{R}$ be the other maximal parabolic of G containing a common 2-Sylow subgroup $\bar{S}$ with $\bar{P}$ and $\bar{M} = O_2(\bar{R})$. Then $\bar{Q}\bar{M} = \bar{S}$ and by [19, (3.2)(3)] $|[\bar{Q}, \bar{M}]| = q^4$. Hence $|[\bar{Q}/\bar{B}, \bar{S}]| = q^2$ which proves (a) by [19, (3.2)(3)].

Assume (b) is false and let $A = V^\alpha \neq V$, $\alpha \in \mathrm{Aut}(S)$ or $\alpha \in \mathrm{Aut}(Q)$. Then, since $A \lhd S$ in the first case, in any case $A \leqslant Q$. Further $P = N_G(Z)$, $Z = C_V(S) = Z(S)$, and, since $\bar{A}$ acts offendingly on V, $\bar{B} \leqslant \bar{A}$ with $|\bar{A}:\bar{B}| = q$. Further, since $\mathfrak{A}(Q) = A \cup V$, $A \lhd P$. We show that Q/A resp. S/A is not isomorphic to $\bar{Q}$ resp. $\bar{S}$ which proves (b).

Let B be the subgroup of A projecting onto $\bar{B}$, $\tilde{P} = O^{2'}(P)$, and $W = C_V(G)[V, \tilde{P}]$. Then VA/WB has order q^2 and is centralized by $\tilde{P}$. ($B \lhd P$, since $A \lhd P$ and $\bar{B} \lhd \bar{P}$!) Since $\Phi(Q/VB) = 1 = \Phi(Q/AW)$, Q/WB is elementary abelian. Hence (a) implies

$$|VQ/WB : [VQ/WB, \tilde{P}]| = q$$

since $|H^1(\tilde{P}/Q, Q/VA)| = q$. Especially

$$VA/WB \cap [Q/WB, \tilde{P}] = AW/BW$$

since $\bar{A}/\bar{B} \leqslant [\bar{Q}/\bar{B}, \tilde{P}]$ by (a) and since $\bar{B} = [\bar{A}, \bar{Q}]$. Thus $|[Q/A, \tilde{P}]| = q^4$ and contains two natural $L_2(q)$-composition factors. Hence $[Q/A, \tilde{P}]$ is elementary abelian of order q^4. Since a maximal elementary abelian subgroup of $\bar{S}$ has order q^3 this proves (b).

(2.6) Lemma. *Let $G = \mathrm{SL}_4(q)$, $q = 2^n > 2$, and W a $\mathrm{GF}(2)G$-module, which has a natural submodule N and quotient W/N equivalent to the orthogonal $\mathrm{GF}(2)G$-module. Suppose $W = \langle C_W(S)^G \rangle$, $S \in \mathrm{Syl}_2(G)$. Then W splits over N.*

Proof. Using Theorem 13 of [5] we only need to show that the dual module V of W also satisfies $V = \langle C_V(S)^G \rangle$. Now V has an orthogonal $\mathrm{GF}(2)G$-submodule U and $V/U \simeq N^*$. Let $P = Q \cdot L$ be the maximal parabolic of G fixing the point $T/U = C_{V/U}(S)$. Then $L' \simeq \mathrm{SL}_3(q)$, Q is the natural L'-module, and $[V/U, Q] = T/U$. Hence for each $x \in Q^\#$ and $t \in T - U$ there exists a $v \in V$ with $tU = [x, v]\,U$. Since x centralizes $[x, v]$ we obtain $[t, x] \leqslant [U, x] \leqslant C_U(Q)$. Thus $[T, Q] \leqslant C_U(Q)$ and by [10]

$$T/C_U(Q) = U/C_U(Q) \times T_0/C_U(Q),$$

where $T_0/C_U(Q)$ is a trivial P'-module. Now by the action of G on the orthogonal GF(2)-module, Q and $C_U(Q)$ are dual L'-modules. Hence $[Q, T_0]=1$ and $T_0 = T_1 \times C_U(Q)$, T_1 a trivial P'-module. This proves (2.6).

(2.7) LEMMA. *Suppose $G = \hat{G}/Z(\hat{G}) \simeq G_2(4)$ with $\mathbb{Z}_2 \simeq Z(\hat{G}) \leqslant \hat{G}'$. Let $N/Z(\hat{G})$ be an elementary abelian normal subgroup of order 4^3 of some maximal parabolic subgroup of $G_2(4)$. Then N is non-abelian.*

Proof. This is obvious, since in a proper covering group of $G_2(4)$ a short root-subgroup lifts to Q_8.

(2.8) PROPOSITION. *Suppose G is an extension of* Sp$(2n, q)$, $q = 2^m \geqslant 8$, *by its natural module V. Suppose there exists an elementary abelian subgroup $A \leqslant G$, $A \cap V = 1$ of order q satisfying $|[V, A]| = q$. Then the extension of G/V by V splits.*

Proof. Proposition (2.8) will be proved by induction on n. For $n = 1$ it is a direct consequence of Gaschütz's theorem. So assume $n > 1$ and (2.8) holds for all $k < n$. We accomplish the proof in several steps.

(1) Without loss A is normalized by some Cartan subgroup H of G. Since A acts as a root group op transvections on V there exists a Cartan subgroup H normalizing VA. Now all involutions of VA lie in $V \cup C_V(A)A$. Hence H normalizes $C_V(A)A$, since it normalizes V. But as $C_V(A) = C_V(VA)$ is normalized by H, H normalizes some complement to $C_V(A)$ in $C_V(A)A$, which proves (1).

Let $\Sigma = A^G$. Then we have:

(2) There exists a subgroup $X \times Y$ of G satisfying:

(i) $X = \langle X \cap \Sigma \rangle \simeq \mathrm{Sp}(2n-2, q)$,

(ii) $A \leqslant Y \simeq L_2(q)$,

(iii) $H \leqslant X \times Y$.

By the induction assumption there exist subgroups X and Y with (i) and $|VA \cap Y| = q$ satisfying $[X, Y] \leqslant V$ and $V = [V, X] \times [V, Y]$. Hence X centralizes $(VY)^\infty \geqslant Y$ and $\langle X, Y \rangle = X \times Y$. Since $VH \cap X \times Y$ is also a Cartan subgroup of G, we may assume $H \leqslant X \times Y$. Now (2) follows readily, if we can show:

(∗) A is the only H-invariant complement to V in VA.

Since all elements of $VA - (V \cup C_V(A)A)$ have order 4, each complement to V in VA must be contained in $C_V(A)A$. Let $\overline{VA} = VA/[V, A]$. Then $[\overline{C_V(A)}, H \cap Y] = 1$. Hence $\bar{A}$ is the only H-invariant complement to $\bar{V}$ in

$\overline{VA}$. It remains to show that A is the only H-invariant complement to $[V, A]$ in $[V, A]A$.

Now $[V, A]A \simeq M_\chi \oplus M_{\chi^{-1}}$ as F_2H-module, where $\chi: H \to F_q^*$ is a surjective homomorphism (in the notation of [16, (2.6)]). Now M_χ and $M_{\chi^{-1}}$ are equivalent F_2H-modules, if and only if χ and χ^{-1} are conjugate under $\mathrm{Gal}(F_q)$. But as $q \geq 8$ this is by [16, (2.6)] not the case. Hence $[V, A]$ and A are the only H-invariant proper subspaces of $[V, A]A$, which proves (*) and (2).

Now let $\bar{G} = G/V$ and $\bar{M} = O_2(C_{\bar{G}}(\bar{A}))$. Then we have:

(3) There exists an HX-invariant subgroup $A \leq M_0 < M$ satisfying $M_0 \cap V = C_V(A)$ and $M_0 V = M$.

Let $\tilde{M} = M/C_V(A)$. Then $\Phi(\tilde{M}) \leq \tilde{V}$ and $\tilde{V}\tilde{A} \leq Z(\tilde{M})$, since all elements of $VA - (V \cup C_V(A)A)$ have order 4. Since X acts in the natural way on $\tilde{M}/\tilde{V}\tilde{A}$ it follows $\Phi(\tilde{M}) = 1$. Since $\tilde{M}/\tilde{V}$ is an indecomposable $\mathrm{GF}(2)X$-module and since by [10] $|H^1(X, \tilde{M}/\tilde{V}\tilde{A})| = q$, we obtain $\tilde{M} = \tilde{V} \times \tilde{M}_0$, $|C_{\tilde{M}_0}(X)| = q$, and $\tilde{M}_0$ HX-invariant.

Now $\tilde{V}\tilde{A} \cap \tilde{M}_0$ is equivalent to $\tilde{A}$ and so to A as F_2H-module. Hence, if $\tilde{V}\tilde{A} \cap \tilde{M}_0 \neq \tilde{A}$, then $[V, A] = \Omega^1(VA \cap M_0)$ is equivalent to A as F_2H-module, which is not the case as shown in the proof of (*).

Next we show:

(4) There exists an HX-invariant subgroup $A \leq M_1 < M_0$ with $M_1 \cap V \leq [V, A]$ and $M_1 V = M$.

Let $\bar{M} = M/[V, A]$ and $H_0 = H \cap Y$. Then $[\overline{C_V(A)}, H_0] = 1$ and H_0 acts like scalar matrices on the "orthogonal" X-module $\bar{M}_0/\overline{C_V(A)}$. Since $q \geq 8$ this shows that $\bar{M}_0$ is elementary abelian. Setting $\bar{M}_1 = [\bar{M}_0, H_0]$, M_1 satisfies (4).

Now the same argument as in the first part of the proof of (3) shows that there is some X-invariant $M_2 < M_1$ satisfying $M_2 \cap V = 1$ and $VM_2 = M$. Since a 2-Sylow subgroup of M_2X is a complement to V in a 2-Sylow subgroup of G, Gaschütz's theorem now implies (2.8).

(2.9) LEMMA. *Suppose G is an extension of* $\mathrm{Sp}(2n, 4)$, *$n \geq 2$, by its natural module V and $P = O^{2'}(N_G(\Omega_1(Z(S))))$, $S \in \mathrm{Syl}_2(G)$. Let $\sigma \in \mathrm{Aut}(P)$ with $V^\sigma \neq V$. Then σ induces an inner automorphism on $P/Q \simeq \mathrm{Sp}(2n-2, 4)$, where $Q = O_2(P)$.*

Proof. Suppose false. Then σ induces a field automorphism on $P/Q = \bar{P}$ and we may assume $\sigma^2 \in Q$. Let $A = [V, V^\sigma]$, $B = V \cap V^\sigma$, and $C = VV^\sigma$. Then the action of P on Q implies $|V^\sigma : B| = 4$. Let $U/B = C_{C/B}(\sigma)$. Then for each $u \in U - B$ the map

$$x \to [x, u], \qquad x \in Q,$$

induces a $GF(2)\bar{P}\langle\sigma\rangle$-isomorphism from Q/C on B/A. But since Q/C and B/A are both natural $\bar{P}$-modules and σ is a field automorphism of $\bar{P}$ we have

$$\mathrm{Hom}_{F_2\bar{P}\langle\sigma\rangle}(Q/C, B/A) \simeq \mathbb{Z}_2,$$

a contradiction to $|U/B| = 4$.

(2.10) Lemma. *Let $G = O^+(2n, 2^m)$ be acting on its natural module V, W a maximal totally singular subspace of V, and $P = U \cdot L = N_G(W)$, with $U = O_2(P)$ and L a Levi complement. Suppose X is a subgroup of G, which acts irreducibly on V and satisfies*:

(1) $O^{2'}(P) \leqslant U(X \cap P)$,

(2) $O_2(X \cap P) = 1$.

Then one of the following holds:

(a) $O^{2'}(X \cap P) \lhd X$,

(b) $m = 1$, $2 \leqslant n \leqslant 4$ *and* $X \simeq (\mathbb{Z}_3 \times \mathbb{Z}_3)\mathbb{Z}_2$, A_7 *or* A_9 *resp.*

Proof. By Hypotheses (1) and (2) $O^{2'}(X \cap P) = Y$ is a complement to U in $O^{2'}(P)$. Now $O^{2'}(P)$ is the split extension of $\mathrm{SL}_n(2^m)$ by the module $\bigwedge^2(N)$, N the natural $\mathrm{SL}_n(2^m)$-module.

Suppose $n \geqslant 3$. Then by [10] either $m = 1$ and $n \leqslant 4$ or all complements are conjugate. In the second case we may assume $Y = O^{2'}(L)$ and so Y contains a long root subgroup of G.

Since Y acts irreducibly on W and V/W either $\langle Y^X \rangle = Y$ or $\langle Y^X \rangle$ is already irreducible on V. Now in the first case (a) holds, while the second case is impossible by [13] and (1) and (2).

So assume $m = 1$ and $n = 3$ or 4. Further, as above we may assume that Y is not conjugate to L in P, whence Y normalizes no complement to W in V. It is now an easy exercise (using for example the list of maximal subgroups in the "atlas") that $X = \langle Y^X \rangle \simeq A_7$ resp. A_9.

So we are finally left with the case $n = 2$. If $m > 1$, then

$$O^{2'2}(P) = O^{2'2}(Y) = Y \simeq L_2(2^m).$$

Hence $Y \lhd G$ and (a) holds. If also $m = 1$, then $P \simeq \mathbb{Z}_2 \times \Sigma_3 \subseteq \Sigma_3 \times \Sigma_3 \simeq G'$. Hence, if $\Sigma_3 \simeq Y \not\lhd G'$, then $X = \langle Y^X \rangle \simeq (\mathbb{Z}_3 \times \mathbb{Z}_3)\langle t \rangle$ with t inverting $O(X)$.

(2.11) Lemma. *Suppose $G = \mathrm{Sp}(2n, 2^m)$ acts on its natural module V. Let $P = Q \cdot L$ be the stabilizer in G of a 1-space V_0 of V, with $Q = O_2(P)$ and Levi complement L. Then the following hold*:

(1) *If* $g \in G$ *such that* V_0^g *is not perpendicular to* V_0, *then* $G = \langle Q, Q^g \rangle = \langle t, Q^g \rangle$ *for each* $\mathrm{GF}(2^m)$*-transvection* $t \in Q^{\#}$.

(2) *If* W *is an irreducible* $\mathrm{GF}(2)G$*-module, then* $W = \langle C_W(Q^g)^P \rangle$.

Proof. Since G acts as a rank 3 permutation group on the 1-spaces of V, P, P^g, $P^{g'}$ are "opposite" maximal parabolics. Hence (2) is just the corollary of [18].

Further, there exist Levi complements L_1 in P and L_2 in P^g normalizing Q and Q^g resp. Q^g and $Q^{g'}$. Let $X = \langle Q, Q^g \rangle$. Then XL_1 is a parabolic containing P and thus $XL_1 = G$. But then $X \trianglelefteq G$ and thus $X = G$. This proves (1).

(2.12) Lemma. *Let* $G = G_2(2^n)$, V *the natural* $\mathrm{GF}(2)G$*-module,* P *a maximal parabolic of* G *stabilizing a one-space in* V, $Q = O_2(P)$, *and* $A \lhd P$ *elementary abelian of order* 2^{3n} *offending on* V. *Then* $G = \langle A, A^g \rangle$ *for* $g \in G$ *with* $C_V(Q) \nleqslant [V, Q^g]$.

Proof. As $V = C_V(Q) \oplus [V, Q^g]$ it is clear that Q and Q^g are "opposite" unipotent radicals. Hence $L = P \cap P^g$ is a Levi complement to Q in P. Now L acts irreducibly on $[V, A]/C_V(Q)$. Thus $[V, A] \cap [V, A^g] = 0$ and $\langle A, A^g \rangle L$ acts irreducibly on V. It is now easy to see that $G = \langle A, A^g \rangle L$. But then $\langle A, A^g \rangle \trianglelefteq G$ and thus $G = \langle A, A^g \rangle$.

(2.13) Lemma. *Let* X *be a solvable group with* $Q_0 = F(X) = O_2(X)$, $S \in \mathrm{Syl}_2(X)$, $1 \neq Z \leqslant \Omega_1(Z(S))$, $V = \langle Z^X \rangle$, $C_S(V) = Q_0$, $J(S) \nleqslant Q_0$, *and* $C_X(V) \leqslant N_X(J(S))$. *Set* $\bar{X} = X/Q_0$ *and* $\bar{Y} = \langle \overline{J(S)}^{\bar{X}} \rangle$. *Then* $\bar{Y}$ *is a direct product of* Σ_3*'s.*

Proof. Let $\tilde{X} = X/C_X(V)$ and, since $Q_0 \leqslant C_X(V)$, consider $\tilde{X}$ as an image of $\bar{X}$. Then by [9, (4.181)] $\tilde{Y}$ is a direct product of Σ_3's. Thus it suffices to show that $\bar{R} = C_{\bar{Y}}(V) = 1$.

Since $C_{\bar{S}}(V) = 1$ we have $\bar{R} \leqslant O(\bar{Y})$. Hence

$$[\bar{R}, \overline{J(S)}] \leqslant O(\bar{Y}) \cap \overline{J(S)} = 1.$$

This implies $\bar{R} \leqslant Z(\bar{Y})$. Let $\bar{a}$ be an involution in $\overline{J(S)}$ with $\bar{Y}_1 = \langle \bar{a}^{\bar{Y}} \rangle \simeq \Sigma_3$. Then $\bar{Y}_1 \simeq \bar{R} \times \Sigma_3$, since a central extension of Σ_3 must split. This implies $\bar{Y} = \bar{Y}_0 \times \bar{R}$ with $\bar{Y}_0$ a direct product of Σ_3's. But as $\bar{Y} = \langle \overline{J(S)}^{\bar{Y}} \rangle$ we obtain $\bar{R} = 1$, which is to show (2.13).

3. Some Reductions

(3.1) *Notation.* We consider in this section the following:

Hypothesis A. G is a finite K-group with

(1) $O_2(G) = Q_0 = F^*(G)$.

(2) There exists some $1 \neq Z \leqslant \Omega_1(Z(T))$, $T \in \mathrm{Syl}_2(G)$, such that for $M = N_G(Z)$ there exists a $Q_1 \trianglelefteq M$ with $Q_0 \leqslant Q_1 \leqslant Q = O_2(M)$ and $C(G, Q_1) = M < G$.

(3.2) Lemma. *In the following two cases M_0 satisfies Hypothesis A with $M = M_0 \cap M_1$ and $Q_i = N_i$, $i = 0, 1$.*

(1) *G satisfies Hypotheses* (a)–(e) *of Theorem* 3.

(2) *G satisfies Hypotheses* (a)–(d) *of Theorem* 3 *and $Z_1 = \Omega_1(Z(N_1)) \leqslant Z(T)$ for some $T \in \mathrm{Syl}_2(M_0 \cap M_1)$.*

Proof. Conditions (a) and (c) of Theorem 3 imply $C(M_0, N_1) = M_0 \cap M_1 = M$. Hence Hypothesis A holds for M_0 in (1).

To prove that Hypothesis A also holds in case (2), it suffices to show that $T \in \mathrm{Syl}_2(M_0)$. So suppose $T < S \in \mathrm{Syl}_2(M_0)$. Then $N_S(T) > T$. But, as $Z_1 = \Omega_1(Z(T))$, $N_S(T) \leqslant N_{M_0}(Z_1) \leqslant M$, a contradiction to $T \in \mathrm{Syl}_2(M)$.

(3.3) Lemma. *Suppose G satisfies Hypothesis A. Let $\bar{G} = G/Q_0$, $\bar{E} = E(\bar{G})$, $\bar{F} = F(\bar{G})$, $V = \langle Z^{EF} \rangle$, $\{\bar{K}_i \mid i = 1, ..., n\}$ the set of components of $\bar{G}$ not in $\bar{M}$, $\{\bar{R}_i \mid i = 1, ..., m\}$ the set of components of $\bar{G}$ contained in $\bar{M}$, $\bar{K} = \prod_{i=1}^{n} \bar{K}_i$, $\bar{R} = \prod_{i=1}^{m} \bar{R}_i$, $\bar{F}_1 = [\bar{F}, J(Q_1)]$, $\bar{F}_2 = C_{\bar{F}}(\overline{J(Q_1)})$.*
Then the following hold:

(1) $C_{Q_1}(V) = Q_0$ *and* $J(Q_1) \nleqslant C(V)$.

(2) *V is an* FF*-module for* $(\bar{K}\bar{F}_1)\,\overline{J(Q_1)}$.

(3) $[RF_2, Q_1] \leqslant Q_0$.

(4) $J(Q_1) \leqslant N(K_i)$ *for* $i = 1, ..., n$.

(5) *If $\bar{F}_1 \neq 1$ then there exists an $A_4 \simeq L \lhd\lhd G$ with $L \nleqslant M$.*

Proof. Because of $C_G(V) \leqslant C_G(Z) \leqslant M$ we have $C_{Q_1}(V) \leqslant O_2(C_G(V))$. Now $EF \leqslant N_G(O_2(C_G(V)))$ and $EF \cap O_2(C_G(V)) \leqslant O_2(C_{EF}(V)) = Q_0$. Hence $[C_{Q_1}(V), \overline{EF}] = 1$ and so $C_{Q_1}(V) = Q_0$, since $\overline{EF} = F^*(\bar{G})$. Since $J(Q_1) \neq J(Q_0)$ this proves (1).

Now F_2 normalizes $J(J(Q_1)Q_0) = J(Q_1)$ and thus $F_2 \leqslant M$. Hence $[RF_2, Q_1] \leqslant Q_1 \cap RF_2 \leqslant O_2(RF_2) = Q_0$.

To prove (2) we need to show $(*)$ $\overline{KF}_1\overline{J(Q_1)} \cap C_{\bar{G}}(V) \leqslant Z(\bar{K})$. As $\bar{F}_1$

is by (2.13) applied to $Q_0FJ(Q_1)$ elementary abelian we have $C_{\overline{KF}_1}(V) \leqslant Z(\bar{K})$, since $[C_{\bar{F}_1}(V), \overline{J(Q_1)}] = 1$. Hence, if $(*)$ is false, then $O_2(\overline{KF}_1 \overline{J(Q_1)}) \neq 1$, which is impossible since $\overline{RF}_2$ normalizes $\overline{KF}_1 \overline{J(Q_1)}$.

Now (4) is [20, (3.1)] and (5) is [2, (8.4)].

(3.4) PROPOSITION. *Suppose that G satisfies Hypothesis A and use the notation of* (3.3). *Let* $Z_i = \langle Z^{K_i} \rangle$, $i = 1, \ldots, n$.

Then the following hold:

(1) $Q_1 \leqslant N(K_i)$ *and* $O_2(K_iQ_1) < Q_1$.

(2) $K_iQ_1/O_2(K_iQ_1) \in \{\mathrm{SL}_n(2^m), \mathrm{Sp}(2n, 2^m), G_2(2^m)\}$, $n \geqslant 2$, $m \geqslant 1$, *and* $\tilde{Z}_i = Z_i/C_{Z_i}(K_iQ_1)$ *is the natural module.*

(3) $K_iQ_1 \cap M/O_2(K_iQ_1)$ *is a maximal parabolic of* $K_iQ_1/O_2(K_iQ_1)$ *with* $O_2(K_iQ_1) < Q_1 \leqslant O_2(K_iQ_1 \cap M)$.

Proof. By (3.3)(4) $J(Q_1) \leqslant N(K_i)$. So let $L_i = K_iJ(Q_1)$ and $\bar{L}_i = L_i/C_{L_i}(Z_i)$. As $O^{2'}(C_{K_i}(Z_i)) = Q_0$ we have $O^{2'}(C_{L_i}(Z_i)) = O_2(L_i)$. Hence $J(Q_1) \nleqslant C_{L_i}(Z_i)$, since $K_i \nleqslant M$. So Z_i is an FF-module for $\bar{L}_i$, $\bar{K}_i = F^*(\bar{L}_i)$ is quasisimple, and $|\bar{L}_i : \bar{K}_i| = 2^{k_i}$. We now accomplish the proof of (3.4) in several steps. We first show:

(I) $\bar{L}_i$ and $\tilde{Z}_i$ satisfy (3.4)(2) for $i = 1, \ldots, n$, where $\tilde{Z}_i = Z_i/C_{Z_i}(L_i)$.

Now Theorem 1 and Sections 8 and 9 of [3] show that one of the following holds:

(1) $\bar{K}_i$ is of Lie type in even characteristic.

(2) $\bar{K}_i \simeq \hat{A}_6$ and Z_i is the module obtained from $\hat{A}_6 \mathrel{\tilde{\subset}} \mathrm{SL}_3(4)$.

(3) $\bar{K}_i \simeq A_n$, $n \geqslant 8$, and $\tilde{Z}_i$ is the natural module. (By the Gaschütz theorem and definition of Z_i we have $\tilde{Z}_i = [\tilde{Z}_i, \bar{L}_i]$.)

(4) $\bar{K}_i \simeq A_n$, $5 \leqslant n \leqslant 8$.

Claim that in all cases:

$(*)$ $\bar{L}_i$ is of Lie type in even characteristic and $\overline{L_i \cap M}$ is contained in a parabolic subgroup of $\bar{L}_i$.

If (1) holds, $(*)$ is a consequence of [12] and the fact that $\overline{L_i \cap M} \leqslant N_{\bar{L}_i}(\overline{J(Q_1)})$ contains a 2-Sylow-subgroup of $\bar{L}_i$ by (3.1)(2). Next it is obvious that (2) and (4) with $n = 7$ contradict $L_i \cap M \leqslant N(Z)$, since in these cases $\bar{L}_i$ has a unique class of offending subgroups. If (4) holds with $n \leqslant 6$, then $\bar{L}_i \simeq L_2(4)$ or $\mathrm{Sp}(4, 2)$ and $(*)$ holds. Moreover, (4) with $n = 8$ and $\tilde{Z}_i$ not the natural A_8-modul leads to $(*)$.

Finally assume (3) holds. If $n = 8$, then $\tilde{Z}_i$ is the module obtained from $A_8 \simeq \Omega^+(6, 2)$ and $\overline{K_i \cap M} = 2^4\Omega^+(4, 2)$. Hence $O_2(\overline{L_i \cap M})$ contains no offending subgroup. Thus $n \geqslant 9$. Then the action of A_n on its natural module shows $\tilde{Z} = C_{\tilde{Z}_i}(\bar{S})$, $S = T \cap L_i$. Hence $\overline{J(Q_1)} \leqslant O_2(N_{\bar{L}_i}(\tilde{Z})) \neq 1$ if and only if $n = 4k$, $k \equiv 1(2)$. Further in this case $|\overline{J(Q_1)}| \leqslant 4$ and it is obvious that $\overline{J(Q_1)}$ contains no offending subgroup.

So $(*)$ holds. We use the description of FF-modules for Lie type groups in even characteristic in [8].

Suppose first $\bar{L}_i \simeq L_n(2^m)$. Then [17, (2.6)(2.7)] imply that Z_i contains only natural or dual non-trivial $\bar{L}_i$-composition factors. Further by [17, (2.6)] $Z_i = C_{Z_i}(\bar{L}_i) \oplus Z_i'$ and all $\bar{L}_i$-composition factors of Z_i' are natural or dual $\mathrm{SL}_n(2^m)$-modules. If now all $\bar{L}_i$-composition factors in Z_i' are equivalent, then by [17, (2.3)] Z_i' is a direct sum of natural modules and $O^{2'}(\bar{P}_i) = C_{\bar{L}_i}(C_{Z_i}(S)) \leqslant \overline{L_i \cap M}$, $\bar{P}_i$ a maximal parabolic of $\bar{L}_i$ with $O^{2'}(\bar{P}_i/O_2(\bar{P}_i)) \simeq \mathrm{SL}_{n-1}(2^m)$. Since $\overline{J(Q_1)} \leqslant O_2(\bar{P}_i)$ contains an offending subgroup it is obvious that Z_i' must be irreducible and thus (I) holds.

So assume Z_i' contains two non-equivalent composition factors. Then by [17, (2.5)]

$$\bar{L}_i \cap M \geqslant O^{2'}(P_i) = C_{\bar{L}_i}(C_{Z_i}(S)) \qquad \text{is the centralizer of a long root subgroup in } \bar{L}_i.$$

But then it is easy to see that $O_2(\overline{L_i \cap M})$ does not contain an offending subgroup on Z_i.

Next suppose $\bar{L}_i$ is orthogonal. Then the argument of [17, (2.9), (2.10)] and the fact that the spin-module for $\Omega^-(6, q)$ is the natural $U_4(q)$-module show that all non-trivial $\bar{L}_i$-composition factors in Z_i are natural and $C_{\bar{L}_i}(C_{Z_i}(S)) \leqslant \overline{L_i \cap M} \leqslant N_{\bar{L}_i}(C_{Z_i}(S))$; the latter being the maximal parabolic with Levi complement orthogonal of dimension $2n-2$. But then $O_2(\overline{L_i \cap M})$ does not contain an offending subgroup on Z_i. By the same reason $\bar{L}_i$ is not unitary. Further, if $\bar{L}_i \simeq G_2(2^m)$, it is easy to see that (I) holds.

So by [8] we are left with the case $\bar{L}_i \simeq \mathrm{Sp}(2n, 2^m)$. If $Z_i' = [Z_i, \bar{L}_i]$ contains an irreducible non-trivial $\bar{L}_i$-composition factor, which is not of dimension $2n$ (over $\mathrm{GF}(2^m)$!), then by [8, 10] $|Z_i'| = 2^{8m}$, $n = 3$, and Z_i' is obtained from an irreducible embedding of $\mathrm{Sp}(6, 2^m)$ in $\Omega^+(8, 2^m)$. Further

$$\overline{L_i \cap M} \subseteq C_{\bar{L}_i}(C_{Z_i}(S)) \simeq 2^{6m}\mathrm{SL}_3(2^m) \subset 2^{6m}\Omega^+(6, 2^m),$$

the latter being a subgroup of $\Omega^+(8, 2^m)$. Hence it is easy to see that $O_2(\overline{L_i \cap M})$ does not contain an offending subgroup.

So all non-trivial $\bar{L}_i$-composition factors in Z_i' are of dimension $2n$. Since (I) obviously holds if $n = 2$, we may assume that they are all equivalent. Now exactly the same argument as in [17, (2.8)] implies that

$$C_{\bar{L}_i}(C_{Z_i}(S)) \leqslant \overline{L_i \cap M} \leqslant N_{\bar{L}_i}(C_{Z_i}(S)),$$

the latter being a maximal parabolic of $\bar{L}_i$ with Levi complement $\mathrm{Sp}(2n-2, 2^m)$. But then $|\bar{W}| = |\bar{A}|\,|C_{\bar{W}}(\bar{A})|$ for each offending subgroup $\bar{A} \leqslant \overline{J(Q_1)}$ and each non-trivial $\bar{L}_i$-composition factor W in Z_i. Hence Z_i contains only one such composition factor and (I) holds in this case too.

Setting $\mathfrak{A}_i = \{A \in \mathfrak{A}(Q_1) \mid A \nleqslant O_2(L_i)\}$ the proof of (I) shows that:

(II) $C_{\bar{L}_i}(C_{Z_i}(S)) \leqslant \overline{L_i \cap M} \leqslant N_{\bar{L}_i}(C_{Z_i}(S))$, the latter being a maximal parabolic of $\bar{L}_i$ with Levi complement isomorphic to $\mathrm{SL}_{n-1}(2^m)$, $\mathrm{Sp}(2n-2, 2^m)$ resp. $\mathrm{SL}_2(2^m)$ in case of $\bar{L}_i \simeq G_2(2^m)$. Further $|A : A \cap O_2(L_i)| = |Z_i : C_{Z_i}(A)|$ for each $A \in \mathfrak{A}_i$.

For the rest of the proof let $\mathfrak{A} = \{A \in \mathfrak{A}(Q_1) \mid A \nleqslant Q_0\}$. An element of $A \in \mathfrak{A}$ is called a minimal offending subgroup (in $\mathfrak{A}$ on V!) if for $B \in \mathfrak{A}$ with $BQ_0 \leqslant AQ_0$ always $BQ_0 = AQ_0$. Then (II) shows:

(III) If $A \in \mathfrak{A}$, then either $(A \cap O_2(L_i))Z_i \in \mathfrak{A}$ or $A \cap O_2(L_i) = A \cap Q_0$ for $i = 1, \ldots, n$. Especially if $A \in \mathfrak{A}$ is a minimal offending subgroup, then either $A \leqslant O_2(L_i)$ or $A \cap O_2(L_i) \leqslant Q_0$.

Next we show:

(IV) (a) Suppose $[K_i, A] \nleqslant Q_0$ and $A \cap O_2(L_i) \leqslant Q_0$ for some $A \in \mathfrak{A}$. Then $Z_i' = [Z_i, K_i] = [V, K_i]$.

(b) If $A \in \mathfrak{A}$ is minimal, then there exists at most one i such that $[K_i, A] \nleqslant Q_0$.

Suppose A satisfies (a). Then

$$|V : C_V(A)| \leqslant |A : A \cap Q_0| = |A : A \cap O_2(L_i)| = |Z_i : C_{Z_i}(A)|$$

by (3.3)(1). Hence $V = Z_i C_V(A)$ and, as $K_i \leqslant \langle A^{K_i} \rangle$, we obtain $Z_i' = [Z_i, K_i] = [V, K_i, K_i] = [V, K_i]$. This proves (a).

To prove (b) assume $[K_i, A] \nleqslant Q_0 \ngeqslant [K_j, A]$ for $i \neq j$ and some minimal $A \in \mathfrak{A}$. Then $[Z_j, A] \leqslant [V, A] \leqslant Z_i'$ because of $V = Z_i' C_V(A)$. Hence

$$Z_j' \leqslant \langle [Z_j, A]^{K_j} \rangle \leqslant Z_i',$$

which is by (I) obviously impossible.

(V) For each $i \leqslant n$ there exists a minimal offending subgroup $A \in \mathfrak{A}$ with $[K_i, A] \nleqslant Q_0$.

Since $[K_i, J(Q_1)] \nleqslant Q_0$ it suffices, to prove (V), to show

(*) For each $A \in \mathfrak{A}$ with $[K_i, A] \nleqslant Q_0$ either $A \cap O_2(L_i) \leqslant Q_0$ or there exists a $B \in \mathfrak{A}$ with $BQ_0 < AQ_0$ and $[K_i, B] \nleqslant Q_0$.

Assume (*) is false for A. Then $B \leqslant O_2(L_i)$ for each $B \in \mathfrak{A}$ with $Q_0 B < Q_0 A$ and, by (II), $D = (A \cap O_2(L_i))Z_i \in \mathfrak{A}$. Let $E \in \mathfrak{A}$ be a minimal offending subgroup with $Q_0E \leqslant Q_0D$. Suppose first $[\bar{F}, E] \neq 1$. Then $|E:E \cap Q_0| = 2$, $\mathbb{Z}_3 \simeq \bar{H} = [\bar{F}, E]$, and $|[V, H]| = 4$. Hence $C = C_A(\bar{H})[V, H] \in \mathfrak{A}$ and $|AQ_0:CQ_0| = 2$. Thus $AQ_0 = C \cdot EQ_0$ and $[K_i, C] \nleqslant Q_0$ which proves $(*)$.

So we may assume $[E, \bar{F}] = 1$. Hence $[K_j, E] \nleqslant Q_0$ for $i \neq j \leqslant n$. If $A \cap O_2(L_j) \nleqslant A \cap O_2(L_i)$, then $(*)$ holds for $Z_j(A \cap O_2(L_j))$. Thus

$$C_A(Z_j) = A \cap O_2(L_j) \leqslant A \cap O_2(L_i) \leqslant C_A(Z_i).$$

Now by (III) either $A \cap O_2(L_j) \leqslant Q_0$ or $Z'_j(A \cap O_2(L_j)) \in \mathfrak{A}$. In the second case $Z_i \leqslant Z'_j C_V(A)$, whence

$$Z'_i \leqslant \langle [Z_i, A]^{K_i} \rangle \leqslant \langle Z_j'^{K_i} \rangle = Z'_j$$

by (IV)(a), which is obviously impossible by (I). Thus $A \cap O_2(L_j) \leqslant Q_0$ and $V = Z'_j C_V(A)$. But then again $Z'_i \leqslant Z'_j$, a contradiction as above. This proves $(*)$ and (V).

(VI) Let $A \in \mathfrak{A}$ be a minimal offending subgroup and $i \leqslant n$ with $[K_i, A] \nleqslant Q_0$. Then

(a) Either $A \leqslant K_i$ or $K_iA/Q_0 \simeq \mathrm{Sp}(4, 2)$ or $G_2(2)$.

(b) In any case $[A, \overline{FR} \prod_{j \neq i} \bar{K}_j] = 1$.

Let $\hat{G} = G/Q_0$. Then by (I) either $\hat{L}_i = \hat{K}_i \times O_2(\hat{L}_i)$ or $\hat{L}_i/O_2(\hat{L}_i) \simeq \Sigma_6$ or $G_2(2)$. By (IV)(b) $[K_j, A] \leqslant Q_0$ for each $j \neq i$. Similarly $[F, A] \leqslant Q_0$. Suppose there is some $a \in \hat{A}^{\#}$ with $a = x \cdot y$, $1 \neq x \in \hat{K}_i$, $1 \neq y \in O_2(\hat{L}_i)$. Then by (3.3) and the above y centralizes $\hat{K}\hat{R}\hat{F}$, a contradiction to $\hat{K}\hat{R}\hat{F} = F^*(\hat{G})$. This shows $A \leqslant K_i$ if $L_i/O_2(L_i)$ is quasisimple. In the other case (a) follows from $A \cap O_2(L_i) \leqslant Q_0$.

Now (b) is an immediate consequence of (IV)(b) and (a).

(VII) For each $i \leqslant n$ we have:

(a) $N_{K_i}(C_{Z_i}(S \cap K_i)) = K_i \cap M = P_i$, with P_i/Q_0 the maximal parabolic of K_i/Q_0 described in (II).

(b) $O_2(L_i) \leqslant Q_0 J(Q_1)$.

If $P_i = N_{K_i}(C_{Z_i}(S \cap K_i))$ then by (II) $O^{2'}(P_i) \leqslant C_{K_i}(Z) \leqslant K_i \cap M$. Further, the second statement about P_i in (a) holds. Suppose $K_i \cap M < P_i$. Then K_i/Q_0 is not isomorphic to A_6 or $G_2(2)'$, whence by (V) and (VI)(a) $(J(Q_1) \cap K_i)Q_0 \lhd P_i$. Since $[O^{2'}(P_i), J(Q_1)] \leqslant J(Q_1) \cap K_i$ the elements of $J(Q_1)$ induce inner automorphisms according to $J(Q_1) \cap K_i$ on K_i/Q_0. Hence $J(Q_1) \leqslant (J(Q_1) \cap K_i)\, O_2(L_i)$ and

$(*)$ $J(Q_1) = (J(Q_1) \cap K_i)(J(Q_1) \cap O_2(L_i))$.

Since $J(J(Q_1)Q_0)=J(Q_1)$ this implies $P_i \leqslant N(J(Q_1)) \leqslant M$, which proves (a).

Now $(*)$ implies (b) in case K_i/Q_0 is not isomorphic to A_6 or $G_2(2)'$. In the latter case either $(J(Q_1) \cap P_i)Q_0 = O_2(P_i)$ or $J(Q_1) \leqslant A(J(Q_1) \cap O_2(L_i))$, $A \in \mathfrak{A}$ a minimal offending subgroup with $[K_i, A] \nleqslant Q_0$. This shows (b) also in this case.

(VIII) $Q_1 \leqslant N(K_i)$ and $O_2(K_iQ_1) < Q_1$ for $i=1, ..., n$.

By (VII)(a) there exists an element $1 \neq x \in K_i \cap M$ of odd order with $x \notin Z(K_i/Q_0)$. Hence $Q_1 = C_{Q_1}(x)[Q_1, x]$ and $[Q_1, x] \leqslant E \leqslant N(K_i)$.

Further, $C_{Q_1}(x) \leqslant N(K_i)$ since Q_1 permutes the components of G/Q_0. This shows $Q_1 \leqslant N(K_i)$. Now $[Q_1, P_i] \leqslant Q_1 \cap K_i \lhd P_i$ so that the second part of (VIII) follows as in the proof of (VII)(b). ($K_i \nleqslant M$!)

Now (VIII) is (3.4)(1). Statement (2) follows from (I) and the fact that Q_1 induces inner automorphisms according to $Q_1 \cap K_i$ on K_i/Q_0, so that either $Q_1 \leqslant K_iO_2(K_iQ_1)$ or $K_iQ_1/O_2(K_iQ_1) \simeq \Sigma_6$ or $G_2(2)$. Statement (3) is now (VII).

(3.5) Corollary. *Suppose G satisfies Hypothesis A. Let K_i be a 2′-component of G not contained in M. Then $Q_1 \leqslant N(K_i)$ and Q_1K_i satisfies Hypothesis A with T and Q_0 replaced by $T \cap Q_1K_i$ and $O_2(K_iQ_1)$.*

(3.6) Corollary. *Suppose G satisfies Hypothesis A and use the notation of* (3.4). *Let $W = \Omega_1(Z(Q_1))$ and $W_i = \langle W^{K_i} \rangle$, $i=1, ..., n$. Then one of the following holds*:

(1) $W_i = Z_i' + C_{W_i}(K_iQ_1)$, $Z_i' = [Z_i, K_i]$;

(2) $|W_i : Z_i' + C_{W_i}(K_iQ_1)| = 2$, $K_iQ_1/O_2(K_iQ_1) \simeq \mathrm{Sp}(2n, 2)$ *and* $|Q_1 : O_2(K_iQ_1)| = 2$.

Proof. Let $A \in \mathfrak{A}$ be a minimal offending subgroup with $[K_i, A] \nleqslant Q_0$. Then $|Z_i' : C_{Z_i'}(A)| = |A : A \cap Q_0|$. Since $A \cap Q_0 \leqslant C(W_i)$ we obtain $W_i = Z_i' + C_{W_i}(A)$. As $O_2(K_iQ_1) < Q_1$ by (3.4)(1) this shows that W_i/Z_i' is a trivial module for $\bar{K}_i = K_iQ_1/O_2(K_iQ_1)$. Hence by [10] either (1) holds or $\bar{K}_i \in \{\mathrm{Sp}(2n, q), G_2(q), L_3(2), L_2(q)\}$. Now it is easy to see that a non-split extension of a trivial $\bar{K}_i$-module by a natural is not an FF-module, except when $\bar{K}_i \simeq \mathrm{Sp}(2n, 2)$ or $L_3(2)$ and the second case contradicts $W_i = Z_i' + C_{W_i}(A)$. So either (1) holds or $\bar{K}_i \simeq \mathrm{Sp}(2n, 2)$ and $|W_i : Z_i' + C_{W_i}(\bar{K}_i)| = 2$ by [10]. If now in the latter case $|Q_1 : O_2(K_iQ_1)| > 2$, then $Q_1 = O_2(K_iQ_1 \cap M)$. But, as $W_i = Z_i' + C_{W_i}(Q_1)$, (2.2) shows that (1) must hold in this case, since Z_i' is K_iQ_1-invariant and $Z_i'/C_{Z_i'}(K_iQ_1)$ is irreducible.

(3.7) COROLLARY. *Let G be a group satisfying the hypothesis of Theorem* 1. *Let $V=\langle Z^{G_0}\rangle$, G_0, Q_0 as in Theorem* 1. *Then the following hold*:

(1) $G_0/Q_0 \in \{\mathrm{SL}_n(2^m), \mathrm{Sp}(2n, 2^m), G_2(2^m)\}$, $n \geqslant 2$, $m \geqslant 1$, and $\tilde{V}=V/C_V(G_0)$ *is the natural module for* G_0/Q_0.

(2) *If* $M_0=M\cap G_0$ *then* M_0/Q_0 *is the maximal parabolic of* G_0/Q_0 *described in* (3.4)(II).

(3) G_0 *satisfies Hypothesis A* (*with* Q_0, Q_1, *and* M).

(4) *If* $W=\langle \Omega_1(Z(Q_1))^{G_0}\rangle$ *then either* $W=C_W(G_0)+V'$, $V'=[V, G_0]$ *or* $|W:C_W(G_0)+V'|=2$, $G_0/Q_0\simeq \mathrm{Sp}(2n, 2)$, *and* $|Q_1:Q_0|=2$.

Proof. The proof of (1) is exactly the same as the proof of (I) in (3.4), since there one only needs the fact that $\overline{L_i\cap M}$ contains a 2-Sylow subgroup of $\bar{L}_i$ and that $O_2(\overline{L_i\cap M})$ contains an offending subgroup.

Now $M_0=N_{G_0}(Z)$. So by (1) $O^{2'}(P)\leqslant M_0$, where P/Q_0 is a maximal parabolic of G_0/Q_0 satisfying (3.4)(II). But by the action of G_0/Q_0 on $\tilde{V}$ we have $J(Q_1)Q_0 \lhd P$, since $J(Q_1)Q_0 \lhd O^{2'}(P)$. Hence $P\leqslant N_{G_0}(J(Q_1))\leqslant M_0$, which proves (2).

Statement (3) is now obvious, since by (2) M_0 is a maximal subgroup of G_0. Hence (4) is (3.6).

(3.8) *Notation.* The group G satisfies *Hypothesis* C, if G is generated by two finite subgroups G_0, G_1 satisfying:

(1) $Q_0=F^*(G_0)=O_2(G_0)$, $Q_1\lhd G_1$, $Q_1\leqslant O_2(G_1)$, and $C_{G_1}(Q_1)\leqslant Q_1$.

(2) $Q_0\leqslant Q_1$ and $(Q_1)_G=1$.

(3) $M=G_0\cap G_1=N_{G_0}(Z)$ for some $1\neq Z\leqslant \Omega_1(Z(S))$, $S\in \mathrm{Syl}_2(G_0)$.

(4) If $V=\langle Z^{G_0}\rangle$, $C=C_V(G_0)$ then $G_0/Q_0\in\{\mathrm{SL}_n(2^m), \mathrm{Sp}(2n, 2^m), G_2(2^m)\}$, $n\geqslant 2$, $m\geqslant 1$, and V/C is the natural G_0/Q_0-module.

(5) $C(G_0, Q_1)=M$.

If G satisfies Hypothesis C it is clear, for example by (3.7)(2), that M/Q_0 is the maximal parabolic of G_0/Q_0 with $O^{2'}(M/O_2(M))\in\{\mathrm{SL}_{n-1}(2^m), \mathrm{Sp}(2n-2, 2^m), \mathrm{SL}_2(2^m)\}$.

If now the hypothesis of either Theorem 1 or Theorem 3 is satisfied or if the hypothesis of Theorem 2 is satisfied and some $2'$-component K_i of G is not contained in M, then (3.2)–(3.7) show the existence of a group G satisfying Hypothesis C. Namely in the latter case we may take $G_0=K_iQ_1$ and $Q_0=O_2(G_0)$ by (3.4), while in the first cases take for G_0 the group denoted by G_0 in Theorem 1. In the situation of Theorem 1 or 2 take for G_1 the amalgamated product of the holomorph of Q_1 with $N_{G_0}(Q_1)$ over Q_1 and for G the free amalgamated product of G_0 and G_1 over Q_1. While in the situation of Theorem 3 we take for G_1 and G the groups denoted by M_1 and G in Theorem 3. Let $Q_1 = N_1$, $G_0 = KQ_1$, where K is the $2'$-complement of M_0 appearing in Theorem 3, and $Q_0 = O_2(G_0)$. Then (3.2) and (3.4) imply that G satisfies Hypothesis C (i.e., (3.4)(1) shows $Q_1 \leqslant N(K)$, so that G_0 is well defined!).

If G satisfies Hypothesis C let $Z_1 = \Omega_1(Z(Q_1))$, $Z_0 = \langle Z_1^{G_0} \rangle$, $C_0 = C_{Z_0}(G_0)$, $\bar{G}_0 = G_0/Q_0$, and $\tilde{Z}_0 = Z_0/C_0$. Then by (3.7)(4) $V' := [V, G_0] = Z_0' := [Z_0, G_0]$ and either $V + C_{Z_0}(G_0) = Z_0$ or $|Z_0 : V + C_{Z_0}(G_0)| = 2 = |Q_1 : Q_0|$ and $G_0/Q_0 \simeq \mathrm{Sp}(2n, 2)$. In the latter case $|\tilde{Z}_0 : \tilde{Z}_0'| = 2$.

(3.9) Lemma. *Suppose G satisfies Hypothesis C. Then $M_G = 1$.*

This is a direct consequence of $(Q_1)_G = 1$ and the structure of G_0.

4. The Graph $\Gamma = \Gamma(G_0, G_1)$

Assume in this section that Hypothesis C is satisfied and use the notation introduced in (3.8). Let $\Gamma = \Gamma(G_0, G_1)$ be the coset graph of G_0 and G_1 in G. Then G is by (3.9) an edge, but not a vertex-transitive automorphism group of Γ. For vertex $\alpha \in \Gamma$ we use the usual notation, namely:

$$G_\alpha, Q_\alpha, Z_\alpha, \bar{G}_\alpha, \Delta(\alpha).$$

Let $Z_\alpha' = [Z_\alpha, \bar{G}_\alpha]$ for $\alpha \sim G_0$ in G. If $\alpha \sim G_0$ in G then we have for $K_\alpha =$ kernel of the action of G_α on $\Delta(\alpha)$: $Q_\alpha = O^{2'}(K_\alpha)$, $[K_\alpha, G_\alpha] \leqslant Q_\alpha$. Let $d(\ ,\ )$ be the usual distance metric on Γ and $d = \mathrm{Min}\{d(\alpha, \beta) \mid Z_\alpha \nleqslant Q_\beta\}$.

A pair (α, β) with $d(\alpha, \beta) = d$ and $Z_\alpha \nleqslant Q_\alpha$ is called a *critical pair*. Then we have:

(4.1) *The following hold*:

(1) $2 \leqslant d \equiv O(2)$,

(2) *If (α, β) is a critical pair, then $G_\alpha \sim G_0 \sim G_\beta$ and $1 \neq [Z_\alpha, Z_\beta] \leqslant Z_\alpha \cap Z_\beta$. Moreover, (β, α) is critical.*

Proof. Let $\alpha, \alpha+1, ..., \beta-1, \beta$ be an arc from α to β. If $\beta - 1 \sim G_0$ in G, then $Q_{\beta-1} \leqslant Q_\beta$ (since $Q_0 \leqslant Q_1$!) and so (α, β) is not critical. So $\beta \sim G_0$ and (3.8) implies $1 \neq [Z_\alpha, Z_\beta] \leqslant Z_\beta$. But by minimality of d also $Z_\beta \leqslant G_\alpha$ but $Z_\beta \nleqslant Q_\alpha \leqslant C_{G_\alpha}(Z_\alpha)$. Hence also (β, α) is critical and (4.1) holds.

We fix for the rest of this section a critical pair (δ_0, δ_d). Let $\delta_0, \delta_1, ..., \delta_d$ be a fixed arc from δ_0 to δ_d. Then we may by edge-transitivity assume $G_0 = G_{\delta_0}$ and $G_1 = G_{\delta_1}$. Thus, to simplify notation, we omit subscript δ for all groups corresponding to vertices of this arc. For vertices in $\Delta(\delta_0) - \delta_1$

resp. $\Delta(\delta_d) - \delta_{d-1}$ we often write δ_{-1} resp. δ_{d+1} and G_{-1} resp. G_{d+1} for the corresponding groups. Let

$$\bigwedge = \bigwedge(\delta_0, \delta_2) := \{\delta_2^g \mid g \in G_0, Z_1 \nleq [Z_0, Q_1^g], Z_2^g \nleq Q_{d-2} \text{ and } Z_d \nleq G_1^g\},$$

$$V_\alpha := Z_\alpha \langle Z_\gamma \mid d(\alpha, \gamma) = 2 \rangle \qquad \text{if} \quad \alpha \sim \delta_0,$$

and

$$V_\beta = \langle Z_\alpha \mid \alpha \in \Delta(\beta) \rangle \qquad \text{if} \quad \beta \sim \delta_1.$$

$\delta_1, \delta_{-1} \in \Delta(\delta_0)$ are called "*opposite*," if $G_0 \cap G_1/Q_0$ and $G_0 \cap G_{-1}/Q_0$ are opposite parabolic subgroups of $\bar{G}_0$.

(4.2) *One of the following holds*:

(1) $\bar{Q}_1 = O_2(\overline{G_0 \cap G_1})$ *and* $Z_1 \leqslant Z(O^{2'}(G_0 \cap G_1))$,

(2) $\bar{Q}_1 = Z(O^{2'}(\overline{G_0 \cap G_1}))$ *and* $\bar{G}_0 \simeq \mathrm{Sp}(2n, 2^m)$,

(3) $\bar{Q}_1 = \bar{Z}_d$ *and* $\bar{G}_0 \simeq G_2(2^m)$.

Proof. This is immediate by (3.8)(4), (4.1)(2), and the action of $\bar{G}_0$ on its natural module.

Next we show:

(4.3) *The following hold*:

(1) $|Z_0 : Z_0 \cap Q_d| = |Z_d : Z_d \cap Q_0|$.

(2) *If* $\bar{G}_0 \simeq \mathrm{SL}_n(2^m)$, $n \geqslant 3$, *then* $C_0' = C_{Z_0'}(\bar{G}_0) = 1$. *In the other cases* $|C_0'| \leqslant 2^m$.

(3) *Either* $C_0' \leqslant [Z_0, Z_d]$ *or* $\bar{G}_0 \simeq \mathrm{Sp}(2n, 2)$ *and* $|Z_0 : Z_0 \cap Q_d| = 2$.

(4) *If* $V_0 \nleq Q_{d-2}$ *for each critical pair* (δ_0, δ_d), *then* $Z_0 Z_2$ *is not normal in* G_0.

(5) *If* $d \geqslant 4$ *and* $V_0 \nleq Q_{d-2}$ *for each critical pair* (δ_0, δ_d) *then* $W_0 := \langle Z_2^{G_0} \rangle \neq Z_0[V_0, Q_0] Z_2$.

Proof. Statement (1) is obvious from the action of $\bar{G}_0$ on $\tilde{Z}_0$. Statement (2) follows from [10] since by hypothesis Z_0' cannot be an indecomposable $L_3(2)$-module of order 16. Statement (3) is (2.1)(a).

To prove (4) we may inductively assume that for each $i \leqslant d/2$ there exists a δ_{-2i} such that $(\delta_{-2i}, \delta_{d-2i})$ is critical. Let $\mu = \delta_{-d+2}$. Then (μ, δ_2) is critical. If now $Z_0 Z_2 \lhd G_0$ there exists a $\lambda \in \Delta(\delta_{-1})$ with $Z_0 Z_2 = Z_0 Z_\lambda$, a contradiction to $[Z_\mu, Z_2] \leqslant [Z_\mu, Z_0 Z_\lambda] = 1$.

To prove (5) let $\tilde{W}_0 = W_0 / Z_0 Z_2$ and suppose $W_0 = Z_0[W_0, Q_0] Z_2$; then

$\tilde{W}_0=[\tilde{W}_0, Q_2]$ and thus $\tilde{W}_0=1$. But then $W_0=Z_0Z_2 \lhd G_0$, a contradiction to (4).

(The proof of (4) and (5) is taken from [11].)

Assume for the rest of this section that $d \geqslant 4$.

(4.4) *Let (δ_0, δ_d) be a critical pair and $\delta_{-1} \in \Delta(\delta_0)$ such that δ_1, δ_{-1} are opposite and $Z_d \nleqslant G_{-1}$. Then $V_{-1} \nleqslant Q_{d-2}$.*

Proof. Suppose (4.4) does not hold for (δ_0, δ_d). If $\bar{G}_0 \simeq \mathrm{SL}_n(2^m)$, $G_2(2^m)$ or if $\bar{G}_0 \simeq \mathrm{Sp}(2n, 2^m)$ and $|Q_1:Q_0|=2^m$, then $[V_{-1}, Z_d]=[Z_0, Z_d] \leqslant Z_0$ and thus $V_{-1} \lhd \langle G_{-1} \cap G_0, Z_d\rangle = G_0$, a contradiction.

Thus $\bar{G}_0 \simeq \mathrm{Sp}(2n, 2^m)$ and $Q_1=O_2(G_0 \cap G_1)$. By (2.11) $G_0=\langle Q_{-1}, Z_d\rangle = \langle Q_{-1}, t\rangle$ for each $t \in Z_d - Q_0$ which induces a transvection on $\tilde{Z}_0$ (over GF(2^m)!). If $[V_{-1}, Z_d \cap Q_0]=1$, then as $Z_0Q_d=C_{Q_{d-1}}(Z_d \cap Q_0)$, $V_{-1} \leqslant Z_0Q_d$ and $[V_{-1}, Z_d] \leqslant Z_0$, a contradiction as before.

Thus $A=[Z_{-2}, Z_d \cap Q_0] \neq 1$ for some $\delta_{-2} \in \Delta(\delta_{-1})$. If $C_0'=1$, then $A \leqslant [Z_0, Z_d]$, a contradiction to $A \dot{\leqslant} Z_{-1} \cap C_{Z_0}(Z_d) \leqslant C_0$, since $Z_d \cap Q_0$ induces transvections on Z'_{-2}. Hence $C_0' \neq 1$. Now (2.1)(b) applied to Z_d' implies $|A| \leqslant 2^m$, while (2.1)(a) applied to Z'_{-2} implies $|A|>2^m$ if $m>1$. Thus $m=1$ and $|C_0'|=2$. If now $|Z_d:Z_d \cap Q_0|>2$, then $[V_{-1}, H] \leqslant A \leqslant C_0$ for $H=Z_{d-1}[Z_d, Q_{d-1}]$, a contradiction since by the above $G_0=\langle G_0 \cap G_{-1}, H\rangle$.

Let $Y=C_{Z_{-2}}(Z_d \cap Q_0)$. Then $Y=Z_{-1}[Z_{-2}, Q_{-1}]$ and $Y \leqslant Z_0Q_d$. Since $G_0=\langle Q_{-1}, Z_d\rangle$ we obtain $YZ_0 \lhd G_0$. If now $\lambda \in \Delta(\delta_{-1})$ with $[Z_\lambda, Z_d \cap Q_0]=1$, then $Z_\lambda Z_0 \lhd G_0$ and, since $[Z_\lambda, O^2(G_0)] \leqslant Z_0$, also $[Z_\lambda, Q_{-1}]Z_0 \lhd G_0$. This implies $Z_0[V_{-1}, Q_{-1}] \lhd G_0$. Especially $[V_{-1}, Q_{-1}] \leqslant V_1 \leqslant C(V_{d-3})$. Since $[Z_{-2}, Q_{-1}] \leqslant C(V_{d-3})$ and thus $|V_{d-3}: V_{d-3} \cap Q_{-2}| \leqslant 2$ we obtain

$$Z_d \cap Q_0 \leqslant Z_{d-2}[V_{d-1}, Q_{d-1}] = Z_{d-2}[V_{d-3}, Q_{d-3}] \leqslant C(Z_{-2}),$$

a contradiction to the choice of δ_{-2}. This proves (4.4).

(4.5) *Suppose (δ_0, δ_d) is critical with $V_d \nleqslant G_1$. Then $\bigwedge = \bigwedge(\delta_0, \delta_2) \neq \varnothing$. Moreover, if $\bar{G}_0$ is not* Sp$(2n, 2)$ *and $|Z_d:Z_d \cap Q_0|=2$, then $\bigwedge \neq \varnothing$ for each critical pair (δ_0, δ_d).*

Proof. Suppose $Z_2^g \leqslant Q_{d-2}$ for each $g \in G_0$ with $Z_d \nleqslant G_1^g$ and δ_1, δ_1^g opposite. Fix such a g and let $\delta_{-1}=\delta_1^g$ and $W=Z_0\langle (Z_2^g)^{G_0 \cap G_{-1}}\rangle$. If $\bar{G}_0 \simeq \mathrm{SL}_n(2^m)$, $G_2(2^m)$ or if $\bar{G}_0 \simeq \mathrm{Sp}(2n, 2^m)$ and $|Q_1:Q_0|=2^m$ then $[W, Z_d]=[Z_0, Z_d] \leqslant Z_0$. Hence $G_0=\langle G_0 \cap G_{-1}, Z_d\rangle \leqslant N(W)$ and $[W, O^2(G_0)] \leqslant Z_0$. This implies $Z_0Z_2=Z_0Z_2^g \lhd G_0$, a contradiction to (4.4) and (4.3)(4).

Thus $\bar{G}_0 \simeq \mathrm{Sp}(2n, q)$, $q=2^m$, and $Q_1 = O_2(G_0 \cap G_1)$. As in the proof of (4.4), (2.1)(b) shows that $A=[W, Z_d \cap Q_0]$ has order q. If now Z_0' is the natural $\bar{G}_0$-module, then $A=[Z_0, Z_d]$, a contradiction to $A \leqslant Z_{-1} \cap C_{Z_0}(Z_d) \leqslant C_0$. Thus $C_0' \neq 1$ and (2.1)(a) applied to the action of $Z_d \cap Q_0$ on Z_2^g implies $q=2=|C_0'|$. Further, if $|Z_d : Z_d \cap Q_0| \neq 2$, then $H = Z_{d-1}[Z_d, Q_{d-1}] \nleqslant Q_0$ and $[W, H] \leqslant A \leqslant C_0$. Since by (2.11) $G_0 = \langle Q_{-1}, H \rangle$ we obtain again $Z_0 Z_2 \lhd G_0$, a contradiction as above. This proves the second part of (4.5).

To prove the first part notice that $A \leqslant Z_1 \cap C_{Z_2}(V_d) \leqslant C_2$ if $V_d \nleqslant G_1$. But as $A \leqslant C_0$ and δ_2, δ_2^g are conjugate in G_0 this implies $A \leqslant C_2^g$, a contradiction to $Z_d \cap Q_0 \nleqslant Q_2^g$.

(4.6) *There exist critical pairs* (δ_2, δ_{d+2}), (δ_0, δ_d), *and* $(\delta_{-2}, \delta_{d-2})$ *with*:

(1) $\delta_0 \in \bigwedge(\delta_2, \delta_4)$,

(2) $\delta_{-2} \in \bigwedge(\delta_0, \delta_2)$.

Proof. Choose a critical pair (δ_2, δ_{d+2}) with $V_{d+1} \nleqslant G_3$. (Exists by (4.4)!) Hence by (4.5) we may pick $\delta_0 \in \bigwedge(\delta_2, \delta_{d+2})$. By definition of $\bigwedge(\delta_2, \delta_{d+2})$ we have $Z_{d+2} \nleqslant G_1$. So we may pick $\delta_{-2} \in \bigwedge(\delta_0, \delta_2)$.

Now we can prove the main result of this section:

(4.7) $d \leqslant 4$.

Proof. Suppose $d > 4$ and choose critical pairs (δ_2, δ_{d+2}), (δ_0, δ_d), and $(\delta_{-2}, \delta_{d-2})$ as in (4.6). Then $1 \neq A = [Z_{-2}, Z_{d-2}] \cap Z_{-1} \nleqslant C_{-2}$ by the action of $\bar{G}_{-2}$ on its natural module $\tilde{Z}_{-2}$. But $A \leqslant Z_{-1} \cap C_{Z_0}(Z_d) \leqslant C_0 \leqslant Z_1$ and thus, since $d > 4$, $A \leqslant Z_1 \cap C_{Z_2}(Z_{d+2}) \leqslant C_2$. By definition of $\bigwedge(\delta_0, \delta_2)$ we have $\delta_{-2} \sim \delta_2$ in G_0. This would imply $A \leqslant C_{-2}$, a contradiction.

5. $d=2$

In this section we carry on with the hypothesis and notation of Section 4. Assume $d=4$ by (4.7) and fix vertices $\delta_{-2}, ..., \delta_6$ of Γ satisfying (4.6). We first show:

(5.1) $\bar{G}_0$ *is not isomorphic to* $G_2(q)$, $q=2^m$.

Proof. Suppose false. Then, as $G_1 \cap G_2 \leqslant N(Z_1)$, $|V_0 : V_0 \cap Q_2| = q^3$ and $V_0 \cap Q_2 \leqslant Z_0 Q_4$. Hence $\tilde{V}_0 = V_0/Z_0$ contains only one non-central chief factor for $\tilde{G}_0 = \langle Z_4^{G_0} \rangle \geqslant O^2(G_0)$ and this is an FF-module.

Let $W_0 = \langle Z_2^{G_0} \rangle$. Then by (4.3)(5) $W_0 \neq Z_0[W_0, Q_0]Z_2$. Hence $[V_0, Q_0, \tilde{G}_0] \leqslant Z_0$ and

$$Z_0[V_0, Q_0] = \prod_{\beta \in \Delta(\delta_0)} [V_\beta, Q_0]Z_0 \leqslant \bigcap_{\beta \in \Delta(\delta_0)} [V_\beta, Q_0]Z_0 \leqslant Z(V_0),$$

since $\tilde{G}_0$ is transitive on $\Delta(\delta_0)$ and centralizes $[V_0, Q_0]Z_0/Z_0$. The three subgroup lemma implies $A = [Z_2, Z_{-2}] \leqslant V_0' \leqslant Z(Q_0)$. Pick by (4.5) $\delta_{-4} \in \bigwedge(\delta_{-2}, \delta_0)$. Then by (2.12) $G_0 = Q_0\langle Z_{-4}, Z_4 \rangle \leqslant C(A)$, a contradiction since $A \nleqslant Z_1$.

(5.2) *Suppose* $\bar{G}_0 \simeq \mathrm{Sp}(2n, q)$, $q = 2^m$. *Then* $Q_1 = O_2(G_0 \cap G_1)$ *and* $|Z_d : Z_d \cap Q_0| = q$ *for each critical pair* (δ_0, δ_d).

Proof. Suppose false and assume first $|Q_1 : Q_0| = q$. Let (δ_0, δ_4) be critical. Then $\langle Q_1, Q_3 \rangle / Q_2 \simeq L_2(q)$, $\delta_1 \sim \delta_3$ in $\langle Q_1, Q_3 \rangle$, and, as $Z_0Z_2 \neq Z_2Z_4$, Z_0Z_4/Z_2 is a natural module for $\langle Q_1, Q_3 \rangle / Q_2$. Hence Z_0Z_4 is elementary abelian, a contradiction to $[Z_0, Z_4] \neq 1$.

Thus $Q_1 = O_2(G_0 \cap G_1)$ and we may by (4.5) assume that $|Z_{-2} : Z_{-2} \cap Q_2| \geqslant q^2 \leqslant |Z_2 : Z_2 \cap Q_{-2}|$. Let $F_{2,0} = [Z_2, Q_1]C_2$ and similarly $F_{\lambda,\mu}$ for $d(\lambda, \mu) = 2$. Then $|[Z_{-2}, F_{2,0}]| \leqslant q$ by (2.1)(b) and so $|F_{2,0} : F_{2,0} \cap Q_{-2}| = q$ and induces a root group of transvections on Z'_{-2}. Hence $|Z_2 : Z_2 \cap Q_{-2}| = q^2$ and so $|Z_\lambda : Z_\lambda \cap Q_\mu| = q$ or q^2 for each critical pair (λ, μ). Now $[F_{-2,0} \cap Q_2, Z_4 \cap Q_0] \leqslant Z_{-1} \cap Z_3 \leqslant C_0 \cap C_2 \leqslant C_{-2}$, since $\delta_2 \sim \delta_{-2}$ in G_0. Hence $Z_4 \cap Q_0 \leqslant C_{Q_{-1}}(F_{-2,0} \cap Q_2) = Z_2Q_{-2}$ and so $Z_2(Z_4 \cap Q_0) \lhd G_2$, since $G_2 = \langle Z_{-2}, Q_3 \rangle$ and $[Z_{-2}, Z_4 \cap Q_0] \leqslant [Z_{-2}, Z_2Q_{-2}] \leqslant Z_2$.

Let $F_2 = Z_2(Z_4 \cap Q_0)$ and $W_2 = \langle F_{4,2}^{G_2} \rangle$. Then, either $W_2 = F_2$ or W_2/F_2 is a non-trivial module for $\bar{G}_2$. In the second case $|Z_4 : Z_4 \cap Q_0| = q^2 = |Z_0 : Z_0 \cap Q_4|$ and so by (2.1)(b) $[Q_2, F_{4,2}] = [Z_0, F_{4,2}] \leqslant Z_1 \cap Z_3 \leqslant C_2$. This implies $[Q_2, W_2] \leqslant C_2$ and $[W_2 \cap Q_0, Z_{-2} \cap Q_2] \leqslant Z_{-1} \cap C_2 \leqslant C_0 \cap C_2$. Hence $W_2 \cap Q_0 \leqslant C_{Q_{-1}}(Z_{-2} \cap Q_2) = Z_2Q_{-2}$, since $\delta_2 \sim \delta_{-2}$ in G_0 and thus $[Z_{-2}, W_2 \cap Q_0] \leqslant Z_2$. But this is a contradiction to $|W_2 : W_2 \cap Q_0| = q$ by definition of W_2 and $|Z_{-2} : Z_{-2} \cap Q_2| = q^2$.

So $F_2 = W_2$ and if $L_2 = \langle Z_4^{G_2} \rangle$, then $|L_2 : L_2 \cap Q_0| \leqslant q$. Let $g \in G_2 \cap G_1$. Then $[F_{4,2}^g, Z_{-2} \cap Q_2] = [Z_2F_{0,2}, Z_{-2} \cap Q_2] = 1$. If $Z_4^g \leqslant Q_0$, then $[Z_4^g, Z_{-2} \cap Q_2] \leqslant Z_{-1} \cap Z_3^g \leqslant C_2$, since by (2.11) $G_2 = \langle Z_{-2}, Q_3^g \rangle$. If $Z_4^g \nleqslant Q_0$, then $[Z_4^g, Z_{-2} \cap Q_2] = [Z_0, Z_4^g] \leqslant Z_1 \cap Z_3^g \leqslant C_2$. Hence if $L_{2,1} = \langle Z_4^{G_2 \cap G_1} \rangle$, then $[L_{2,1} \cap Q_0, Z_{-2} \cap Q_2] \leqslant Z_{-1} \cap C_2 \leqslant C_0 \cap C_2 \leqslant C_{-2}$ and so $L_{2,1} \cap Q_0 \leqslant Z_2Q_{-2}$ and $[L_{2,1} \cap Q_0, Z_{-2}] \leqslant Z_2$.

Now by (2.2) there exists a non-trivial $\bar{G}_2$-top composition factor $\tilde{L}_2$ of L_2/F_2. Since $\tilde{L}_2 = \langle \tilde{Z}_4^{G_2} \rangle$ and $[\tilde{Z}_4, Q_3] = 1$, (2.11) implies $\tilde{L}_2 = \tilde{L}_{2,1}$.

($\bar{Q}_1$ and $\bar{Q}_3$ are "opposite" unipotent radicals in $\bar{G}_2$. Further $\overline{G_1 \cap G_2} = \bar{Q}_1(\overline{G_1 \cap G_2 \cap G_3})$ and $\overline{G_2 \cap G_3} = \bar{Q}_3(\overline{G_1 \cap G_2 \cap G_3})$.) Hence Z_{-2} centralizes a subgroup of index q in $\tilde{L}_2$, a contradiction to $|Z_{-2}:Z_{-2} \cap Q_2| = q^2$.

(5.3) *$\bar{G}_0$ is not isomorphic to* $\mathrm{Sp}(2n, q)$, $q = 2^m$.

Proof. Suppose false and let $F_{\lambda,\mu}$ be defined as in (5.2). Then by (5.2) $[F_{-2,0}, F_{4,2}] \leqslant Z_{-1} \cap Z_3 \leqslant C_0 \cap C_2 \leqslant C_{-2}$, since $F_{-2,0} \leqslant Q_2$ and $F_{4,2} \leqslant Q_0$. Hence $[Z_{-2}, F_{4,2}] \leqslant [Z_{-2}, Z_2] \leqslant Z_0$ and $Z_2 F_{4,2} \lhd G_2 = \langle Z_{-2}, Q_3 \rangle$. Similarly $Z_0 F_{-2,0} = Z_0 F_{2,0} \lhd G_0$. Let $L_2 = \langle Z_4^{G_2} \rangle$. Then $|L_2:L_2 \cap Q_0| = q$ and $[L_2 \cap Q_0, F_{-2,0}] = [L_2 \cap Q_0, F_{2,0}] = 1$. Hence $L_2 \cap Q_0 \leqslant Z_2 Q_{-2}$ and Z_{-2} centralizes a subgroup of index q in $\tilde{L}_2 = L_2/Z_2 F_{4,2}$. Now, as $Z_2 Z_4 \neq Z_2 Z_0$, $\tilde{L}_2$ is a non-trivial FF-module for $\bar{G}_2$. Hence by [8] $\hat{L}_2 = \tilde{L}_2/\tilde{T}_2$ is equivalent to Z_2/C_2 as $\bar{G}_2$-module where $\tilde{T}_2 = C_{\tilde{L}_2}(\bar{G}_2)$. Now $T_2 \leqslant L_2 \cap Q_0$ since $|[Z_{-2}, \tilde{L}_2/\tilde{T}_2]| = q$. Hence $T_2 \leqslant Z(L_2)$ since $L_2 = \langle Z_0^{G_2} \rangle$ by (4.6). Now, considering $\hat{L}_2$ as symplectic space, $\hat{Z}_0 \hat{Z}_4$ is a hyperbolic plane. Since $\bar{G}_2$ is transitive on the hyperbolic pairs of $\hat{L}_2$ and since L_2 is generated by involutions, this implies $\Phi(L_2) = L_2' \leqslant C_2$, as $[Z_0, Z_4] \leqslant C_2$.

Set $L_0 = \langle Z_2^{G_0} \rangle$ and pick $t \in Z_4 - F_{4,2}$. Then, since $L_0 \cap Q_2 \leqslant Z_0 Q_4$ as for L_2, it follows that

$$[L_0, t, t] \leqslant \Phi(L_0) \cap [L_0 \cap Q_2, Z_4] \leqslant [Z_{-2}, Z_2] \cap [Z_0, Z_4] = 1,$$

since $[Z_{-2}, Z_2] \cap [Z_0, Z_4] \leqslant C_0 \cap C_2$ and $\delta_2 \sim \delta_{-2}$ in G_0. Hence $[L_0, t] \leqslant C_Q(F_{4,2}\langle t \rangle) \leqslant Q_4$ and thus by the action of $\bar{G}_2$ on $\hat{L}_2$

$$[L_0, \hat{t}] \leqslant [Z_{-2}, \hat{t}] \leqslant Z_0 \cap Q_4 = 1,$$

a contradiction to $L_0 \nleqslant Q_2$.

(5.4) *$\bar{G}_0 \simeq L_4(2) \simeq \Omega^+(6, 2)$. Further, if $N_0 = Q_0 \cap O^2(G_0)$, then $\mathbb{Z}_2 \simeq \Phi(N_0) \leqslant C_0$ and $N_0/\Phi(N_0)$ is an indecomposable $\bar{G}_0$-module, which is the extension of a natural $\Omega^+(6, 2)$-module by the natural $L_4(2)$-module $Z_0' \Phi(N_0)/\Phi(N_0)$. Moreover, $Q_0 = N_0 C_{Q_0}(O^2(G_0))$.*

Proof. Suppose false. Then by (5.1), (5.3) $\bar{G}_0 \simeq L_m(q)$, $q = 2^n$. Let $W_0 = \langle Z_2^{G_0} \rangle$ and $W_2 = \langle Z_0^{G_2} \rangle$. Then, as

$$[Q_0, Z_2] = [Z_{-2}, Z_2] \leqslant Z_{-1} \cap Z_1 \leqslant C_0,$$

we have $W_0' \leqslant [Q_0, W_0] \leqslant C_0$. By symmetry we may assume

$$|W_0 : W_0 \cap Q_2| \leqslant |W_2 : W_2 \cap Q_0|.$$

Hence $\tilde{W}_0 = W_0/Z_0$ is an FF-module for $\bar{G}_0$ with offending subgroup $\tilde{W}_2$, since $[W_0 \cap Q_2, W_2] \leqslant C_2 \leqslant Z_0$. Hence by [8] and the remark following the statement of the theorem in [8] one of the following holds:

(a) All non-trivial $\bar{G}_0$-composition factors in $\tilde{W}_0$ are equivalent natural modules.

(b) $m \geqslant 4$ and $\tilde{W}_0/\tilde{T}_0$ is the symmetric square of a natural module, where $T_0 = C_{\tilde{W}_0}(\bar{G}_0)$. Further $Q_1 = Q_0 W_2$.

If now $m \geqslant 3$ in case (a), then [17, (2.3), (2.6)] implies $\tilde{W}_0 = \tilde{T}_0 \times \tilde{W}'_0$, $\tilde{T}_0 = C_{\tilde{W}_0}(\bar{G}_0)$, $\tilde{W}'_0 = [\tilde{W}_0, \bar{G}_0]$ is the direct sum of equivalent natural modules. Since $m \geqslant 3$ and $\Phi(W'_0) \leqslant C_0$ we obtain $\Phi(W'_0) = 1$, since these modules are not self-dual. Further $[T_0, W'_0] = 1$ and so $\Phi(W_0) = 1$, since $W_0 = Z_2 W'_0$, a contradiction to $\delta_2 \sim \delta_{-2}$ in G_0.

So $m = 2$ in case (a). Hence $\tilde{W}_0/\tilde{T}_0$ is a natural $\bar{G}_0$-module and $\tilde{W}_0 = \tilde{T}_0 \tilde{Z}_{-2} \tilde{Z}_2$. Now, as $[W_0 \cap Q_2, W_2] \leqslant C_2 \leqslant Z_0$, $T_0 = W_0 \cap Q_2 \cap Q_{-2} = Z(W_0)$. Further $\Phi(W_0) = [Z_2, Z_{-2}]$ and $Q_1 = Q_0 W_2 = Q_0 Z_4$, since $W_2 = T_2 Z_0 Z_4$. Pick $t \in Z_4 - Q_0$. Then

$$[Z_{-2}, t, t] \leqslant \Phi(W_0) \cap [W_0 \cap Q_2, t] \leqslant [Z_2, Z_{-2}] \cap [Z_0, Z_4] = 1$$

as in (5.3). Hence $T_2[Z_{-2}, t] \leqslant C_{W_2}(t) = W_2 \cap Q_4$, a contradiction since by the action of $L_2(q)$ on its natural module $W_2 = T_2[Z_{-2}, t] Z_4$.

So (b) holds. As $\tilde{W}_0$ is an FF-module, (2.3) and [10] show that in any case $\tilde{W}_0 = \tilde{T}_0 \oplus \tilde{W}'_0$, $\tilde{W}'_0 = [\tilde{W}_0, \bar{G}_0]$. Further $m = 4$, since $\tilde{W}'_0$ must be self-dual. Let $F_0 = C_{Q_0}(O^2(G_0))$ and claim (*) $Q_0 = W'_0 F_0$.

Now by (b) $Q_0 = (Q_0 \cap Q_2) W'_0$ and $[Q_0, W_2] \leqslant C_2[W'_0, W_2]$. Let $\bar{Q}_0 = Q_0/\Phi(Q_0) C_0$. Then again (2.3) and [10] imply $\bar{Q}_0 = \bar{W}'_0 \times \bar{C}^0$, $\bar{C}^0 = C_{\bar{Q}_0}(\bar{G}_0)$. As $[W_0, Q_0] \leqslant C_0$, we have $\Phi(Q_0) \leqslant C_0 \Phi(C^0)$ and thus $C^0 \leqslant F^0$ which proves (*).

As $[W'_0, F_0] \leqslant C_0$ the three-subgroup lemma implies $W'_0 = C_0[W'_0, O^2(G_0)] \leqslant C(F_0)$. Further, since $W_2 \leqslant Q_0 O^2(G_0)$, $[F_0, W_2] \leqslant F'_0 \cap C_2[W_0, W_2] \leqslant C_0 \leqslant Z_1$. Hence $F_0 \leqslant Q_2$ and so $F_0 \leqslant C_{Q_0}(W_0)$. Especially, if $L_0 = F_0 \cap W_0$, then $T_0 = Z_0 L_0 \leqslant Z(Q_0)$. Now $G_0 \cap G_1$ acts transitively on $[\tilde{W}'_0, Q_1]^{\#}$. Hence $[W'_0, Q_1]$ is elementary abelian and so $T_0[W'_0, Q_0] = Z_0 Z_2 [W'_0, Q_0]$ is also elementary abelian. This implies $L_0 \leqslant \Omega_1(Z(Q_1))$ and thus $T_0 = Z_0$ and $W_0 = W'_0$.

Let $\hat{W}_0 = W_0/C_0$ and suppose $q \geqslant 4$. Then $\hat{Z}_2 = C_{\hat{W}_0}(Q_1)$ and so by [10] $\hat{Z}_2 = \hat{Z}_1 \times [\hat{Z}_2, G_0 \cap G_1]$. Hence $|C_{\hat{W}_0}(S)| = q^2$ and (2.6) implies that the extension of $\hat{W}_0/\hat{Z}_0$ by $\hat{Z}_0$ splits. But then $\hat{Z}_2 = C_{\hat{W}_0}(Q_1) = [\hat{W}_0, Q_1]$ and thus $Z_2 \leqslant Z_1 Q'_1$. On the other hand, as $Q_1 = (W_0 * F_0) W_2$ and $W'_2 \leqslant C_2 \leqslant Z_1$, it follows that

$$Z_0 \cap Q'_1 \leqslant Z_0 \cap Z_1[W_0, W_2] \leqslant Z_0 \cap Z_2 = Z_1,$$

a contradiction to $\delta_0 \sim \delta_2$ in G_1. Hence $q = 2$.

This argument also shows that the extension of W_0/Z_0 by Z_0/C_0 does not split if $q=2$. Now, as $W_2 \cap Q_0 = Z_0 Z_2 \leqslant W_0$ and $Q_0/W_0 \leqslant Z(G_0/W_0)$, it follows that $N_0 \leqslant W_0$. By the action of G_0 on W_0/Z_0 we have

$$\Phi(N_0) = N_0' \leqslant W_0' = [Z_2, Z_{-2}] \simeq \mathbb{Z}_2.$$

Thus if $\bar{N}_0 = N_0/\Phi(N_0)$ an easy application of (2.3) yields $\bar{N}_0 = [\bar{N}_0, G_0] \times \overline{C_0 \cap N_0}$. As $W_2' = [Z_0, Z_4] \leqslant [N_0, G_0]$ we obtain $\overline{C_0 \cap N_0} = 1$ and (5.4) holds. (The last statement follows from (∗)!)

6. Proof of Theorems 1 and 2

In this section we carry on with the hypothesis and notation of Section 4. Thus by (4.1) and Section 5 $d=2$, except in case (5.4).

(6.1) *Suppose* $\bar{G}_0 \simeq L_n(q)$, $n \geqslant 2$, $q = 2^m \geqslant 2$. *Let* $O^2(G_0) = R_0$ *and* $N_0 = Q_0 \cap R_0$. *Then one of the following holds*:

(1) $n \geqslant 3$, $\Phi(N_0) = 1$, *and* N_0 *is the natural module for* $R_0/N_0 \simeq \mathrm{SL}_n(q)$.

(2) $n=2$, $\Phi(N_0)=1$, $N_0 = [N_0, R_0]$, *and* $N_0/N_0 \cap C_0$ *is the natural module for* $R_0/N_0 \simeq \mathrm{SL}_2(q)$.

(3) $R_0 \simeq 2^4 \hat{A}_8$ *or* $4^3\mathrm{SL}_3(4)$, *where* $\hat{A}_8$ *resp.* $\mathrm{SL}_3(4)$ *are perfect central extensions of* A_8 *resp.* $\mathrm{SL}_3(4)$ *by an elementary* 2-*group of order* 2 *resp.* $\leqslant 4$.

(4) (5.4) *holds.*

Proof. Suppose case (4) does not hold. Then $d=2$. Further, if $n=2$, then it is well known and easy to see that (2) holds. (See [14].) So assume $n \geqslant 3$. Then by (4.3) $Z_0' = [Z_0, G_0]$ is the natural $\bar{G}_0$-module. Hence

$$(*) \quad Z_1' := [Z_0', Z_2'] = [Z_0', Q_1] = [Z_2', Q_1]$$

and $[Q_0, R_0] \leqslant Z_0'$ since $R_0 \leqslant \langle Z_2'^{G_0} \rangle$. Let $C^0 = C_{Q_0}(R_0)$. Then $Q_0 = Z_0' \times C^0$, since $Z_0' \cap \Phi(Q_0) = 1$ and since by (∗) $|[Q_0, Z_2']| = 2$ in case $q=2$, $n=3$. This shows that, except in case $n=3$, $q \leqslant 4$ or $n=4$, $q=2$, (1) holds. (Since the multiplier of $\bar{G}_0$ is trivial!)

In these cases let $L_0 = \langle Z_2^{G_0} \rangle$ and $\bar{L}_0 = L_0/Z_0'$. Then by (∗) $[Z_2, Q_1] = Z_1' \leqslant Z_0'$, so that $\bar{Z}_2 \leqslant Z(\bar{Q}_1)$. Especially $\langle \bar{Z}_2^{G_0 \cap G_1} \rangle$ is elementary abelian. This shows that in case $n=3$, $q=2$, $\bar{L}_0$ is not a central product of $\mathrm{SL}_2(7)$ with some 2-group. Hence in this case also (1) holds. Further in case $n=3$, $q=4$, $Z(\bar{L}_0) \cap O^2(\bar{L}_0)$ must be elementary abelian. Hence in the remaining cases (3) holds.

(6.2) *Suppose* $\bar{G}_0 \simeq G_2(q)$ *and let* $R_0 = O^2(G_0)$, $N_0 = Q_0 \cap R_0$. *Then*

(a) $\Phi(N_0) = 1$, $N_0/N_0 \cap C_0$ *is the natural module for* $R_0/N_0 \simeq G_2(q)'$, *and* $N_0 = [N_0, R_0]$. *Further* $Q_0Q_2 = (Q_0 \cap Q_2)Z_0Z_2$.

(b) *If* $q \geqslant 4$, *then* $|Q_1 : Q_0| = q^3$.

Proof. Because of $|Z_2Q_0 : Q_0| = q^3$ and $Z_2 \leqslant Z(Q_2)$ we have $Q_0Q_2 = Q_0Z_2 = Q_2Z_0$. Since $R_0 \leqslant \langle Z_2^{G_0} \rangle$ this implies $[Q_0, R_0] \leqslant Z_0'$. Now the multiplier of $G_2(q)$ is trivial, except in case $q = 4$. This shows that (6.2)(a) holds when $q \neq 4$.

Now $[Q_0, Z_2] = [Z_0', Z_2] \leqslant Z_0'$. Hence N_0Z_2/Z_0' is abelian, since $N_0 \leqslant \langle Z_2^G \rangle$. But then (2.7) shows that R_0/Z_0' is not a proper covering group of $G_2(4)$. This shows that (6.2)(a) also holds in case $q = 4$.

To prove (b) we may replace G_1 by $\langle G_0 \cap G_1, \sigma \rangle = \hat{G}_1$, where $\sigma \in G_1$ with $\delta_0^\sigma = \delta_2$, and G by $\hat{G}/Q$, where $\hat{G} = \langle G_0, \hat{G}_1 \rangle$, $Q = (Q_1)_{\hat{G}}$, since also $\hat{G}/Q$ satisfies Hypothesis C for $d = 2$ and (b) holds for G if it holds for $\hat{G}/Q$. Hence we may assume $G = \langle G_0, \sigma \rangle$. But then $\Phi(Q_0) = \Phi(Q_0 \cap Q_2) = \Phi(Q_2) = 1$ and so the hypothesis of (2.5)(b) is satisfied, since Q_0 is an FF-module for G_0. This implies (b).

(6.3) *Suppose* $\bar{G}_0 \simeq \mathrm{Sp}(2n, q)$, $q = 2^m$, $n \geqslant 2$, *and* $Q_1 = Q = O_2(M)$. *Set* $R_0 = O^2(G_0)$, $N_0 = Q_0 \cap R_0$. *Then one of the following holds*:

(1) $R_0/N_0 \simeq \mathrm{Sp}(6, q)$, $N_0/Z(N_0)$ *is the* 8-*dimensional irreducible orthogonal* $\mathrm{Sp}(6, q)$-*module.* $\Phi(N_0) \leqslant Z(N_0)$ *has order* q *and* $[Z(N_0), R_0]$ *is the natural* $O_7(q)$-*module for* R_0.

(2) $R_0/N_0 \simeq \mathrm{Sp}(2n, 2)'$, $\Phi(N_0) = 1$, *and* $N_0/Z(R_0)$ *is the natural* $\mathrm{Sp}(2n, 2)'$-*module.*

(3) $R_0/N_0 \simeq \mathrm{Sp}(2n, 4)$, $\Phi(N_0) = 1$, *and* $N_0/Z(R_0)$ *is the natural* $\mathrm{Sp}(2n, 4)$-*module.*

Further, the extension of R_0/N_0 *by* N_0 *does not split.*

Further in (2) *and* (3) $N_0 = [N_0, R_0]$, $Q_i = (Q_i \cap Q_j)Z_i$, *and* $|Q_iQ_j/Q_j| = 2$ *resp.* 4 *for* $\{i, j\} = \{0, 2\}$.

Proof. Let $V_0 = [N_0, R_0] = [Q_0, R_0]$. If $V_0 \nleqslant Z_0$ then the proof of (5.13) of [11] shows that (1) holds.[1] So we may assume $V_0 \leqslant Z_0$. If now $V_0 \neq N_0$ then R_0/V_0 is a non-split central extension of $\mathrm{Sp}(2n, q)$. Hence by [7, Tables 3 and 4, p. 20] $q = 2$ and $n = 2$ or 3. So we may assume $N_0 \neq V_0Z(R_0)$, since otherwise (2) holds. Thus $N_0/Z(R_0)$ is an indecomposable module for R_0.

[1] (5.13) of [11] proves that (6.3)(1) holds if $V_0 \nleqslant Z_0$ under the hypothesis: G_0 as above, $d = 2$, $G_1 = \mathrm{Hol}(S)$, $S \in \mathrm{Syl}_2(G_0)$, and $G = G_0 *_S G_1$. Although this hypothesis is slightly different from ours, the proof goes through.

Suppose first $n=2$. Then $R_0/V_0 \simeq \mathrm{SL}_2(9)$. Since Z'_2 induces a transvection on Z'_0 and since $[Z'_2, Q_1] \leqslant R_0$ and $|Z'_1 : [Z'_2, Q_1]| = 2$ we have $R_0 Z'_2/N_0 \simeq \mathrm{Sp}(4, 2)$. As R_0/V_0 contains only one involution an easy computation shows that $R_0 Z'_2 \cap G_1/V_0 \simeq \mathbb{Z}_4 * \mathrm{SL}_2(3)$, which is impossible since $\Phi(Z'_2) = 1$.

So $n=3$. Let $L_0 = R_0 Z'_2$. Since by the structure of $\widehat{\mathrm{Sp}(6, 2)}$ we have $Q_1 \cap R_0/V_0 \simeq \mathbb{Z}_4 * 2^{1+4}$ the same argument as above implies $R_0 \neq L_0$. Hence $L_0/V_0 \simeq \widehat{\mathrm{Sp}(6, 2)} * \mathbb{Z}_4$ since $[R_0, Z'_2] \nleqslant N_0$. Let $t \in L_0$ with $\langle t^2 \rangle V_0 = N_0 \geqslant [R_0, t]$. Then $[R_0, t] \leqslant V_0$ and $[V_0, t] = 1$. Hence $t^2 \in Z(R_0)$, a contradiction to $N_0 \neq V_0 Z(R_0)$.

Let $C^0 = C_{Q_0}(R_0)$, $Z'_1 = [Z'_0, Z'_2]$, and suppose $q \geqslant 4$. (If $q=2$ (2) holds by what we have shown!) Suppose there is a R_0-invariant subgroup $V_0 < W \leqslant Q_0$ such that $W/Z(R_0)$ is an indecomposable R_0-module with $[W, R_0] \leqslant V_0$. Then $|[W, Z'_2]| = q\,|Z(R_0)|$, since Z'_2 induces a transvection group on V_0. Hence $W \leqslant V_0 Q_2$ by the action of Q_1 on Z'_2. This implies $|W : C_W(Z'_2)| = q$ and thus $W = V_0 C_W(R_0)$, since by $q > 2$ $\mathrm{Sp}(2n, q)$ is generated by $2n$-transvection groups, a contradiction.

So no such W exists, which shows $Q_0 = V_0 C^0$, $C^0 = C_{Q_0}(R_0)$. Suppose $Q_0 \nleqslant V_0 Q_2$. Then $C^0 \nleqslant V_0 Q_2$ and, as $Q_1 = V_0 C^0 (Q_1 \cap R_0)$ and $[Z'_2, C^0] \leqslant Z'_2 \cap C^0 \leqslant C(Q_1 \cap R_0)$, (2.1)(b) implies $Q_1 \cap R_0 \leqslant Q_2 V_0$ or $[Z'_2, C^0, C^0] = [Z'_2, Q_1, Q_1] = [Z'_2, Q_1 \cap R_0, Q_1 \cap R_0]$. In the first case $Q_1 = Q_0 Q_2 = V_0 C^0 Q_2$ and

$$Z'_1 \leqslant [Z'_2, Q_1] = [Z'_2, Q_0] = [Z_2, C^0] \leqslant C^0$$

which is not the case. In the second case

$$[Z'_2, Q_1 \cap R_0, Q_1 \cap R_0] \leqslant [Z'_2 \cap Z_0, Q_1 \cap R_0] \cap C^0 \leqslant C'_0.$$

Hence $[Z'_2, Q_1 \cap R_0] \leqslant C_{Z_0}(Q_1 \cap R_0) \cap Z'_2 \leqslant Z_1 \cap Z'_2 = Z'_1$, which is impossible if $Q_1 \cap R_0 \nleqslant Q_2 V_0$.

This implies $Q_0 \leqslant V_0 Q_2$, $Q_2 \leqslant V_2 Q_0$, and $Q_i = (Q_i \cap Q_j) V_i$ for $\{i, j\} = \{0, 2\}$. Hence $\Phi(Q_0) = \Phi(Q_0 \cap Q_2) = \Phi(Q_2) \lhd R = \langle G_0, \sigma \rangle$, $\sigma \in G_1$, with $\delta_0^\sigma = \delta_2$. Let $N = (Q_1)_R$ and $\bar{R} = R/N$. Then $N \cap Q_0 \leqslant C^0$, since $Q_0 = V_0 C^0$ and $V_0/V_0 \cap C^0$ is not centralized by V_2. Thus

$$\bar{Q}_0 \cap \bar{Q}_2 = C_{\bar{Q}_i}(\bar{Q}_j)$$

and $|\bar{Q}_i : \bar{Q}_i \cap \bar{Q}_j| = q$, $\bar{Q}_i = \bar{V}_i(\bar{Q}_i \cap \bar{Q}_j)$, and $\Phi(\bar{Q}_i) = 1$ for $\{i, j\} = \{0, 2\}$. Hence all involutions of $\bar{Q}_0 \bar{Q}_2$ lie in $\bar{Q}_0 \cup \bar{Q}_2$. If now the extensions of R_0/V_0 by V_0 split, then $\bar{G}_0 = \bar{Q}_0 \cdot \bar{X}$, $\bar{X} \simeq \mathrm{Sp}(2n, q)$. Hence $\bar{Q}_2 = (\bar{Q}_2 \bar{Q}_0 \cap \bar{X})(\bar{Q}_0 \cap \bar{Q}_2)$ and thus $\bar{Q}_2 \leqslant Z_2(\bar{Q}_1)$, which contradicts the action of $\bar{Q}_1$ on $\bar{Q}_2$. ($\bar{Q}_2$ contains a natural module for $\bar{G}_2$ and thus $|[\bar{Q}_2, \bar{Q}_1, \bar{Q}_1]| \geqslant q$.)

Thus the extensions of R_0/V_0 by V_0 and $\bar{G}_0/\bar{Q}_0$ by $\bar{Q}_0$ do not split. Let $\bar{H}$ be a Cartan subgroup of $\bar{G}_0$ normalizing $\bar{Q}_0\bar{Q}_2$. Then $\bar{H}$ normalizes $\bar{Q}_2$ and so there exists an $\bar{H}$-invariant complement $\bar{A}$ to $\bar{Q}_0 \cap \bar{Q}_2$ in $\bar{Q}_2$. Hence by (2.8) the extension of $\bar{G}_0/\bar{Q}_0$ by $\bar{Q}_0/C_{\bar{Q}_0}(\bar{G}_0)$ splits if $q \geqslant 8$, a contradiction to the above.

This shows that $q \leqslant 4$ and (3) holds if $q = 4$.

(6.4) *Suppose* $\bar{G}_0 \simeq \mathrm{Sp}(2n, q)$, $q = 2^m$, $n \geqslant 2$, *and* $Q_1 \neq Q$. *Then* $|Q_1 : Q_0| = q$ *and if* $R_0 = O^2(G_0)$, $N_0 = Q_0 \cap R_0$ *the following hold*:

(1) $\Phi(N_0) = 1$, $N_0 = [N_0, R_0]$, *and* $N_0/Z(R_0)$ *is the natural module for* $R_0/N_0 \simeq \mathrm{Sp}(2n, q)'$.

(2) *If* $q \geqslant 8$, *then the extension of* R_0/N_0 *by* N_0 *splits.*

Proof. Since $Q_0 < Q_1 < Q$ and $Q_1 < G_0 \cap G_1$ it is obvious that $|Q_1 : Q_0| = q$, $Q_1 = Q_0 Z_2 = Q_2 Z_0 = Q_0 Q_2$, and $[Q_0, Z_2] = [Z_0, Z_2] \leqslant Z_0 \cap Z_2$. Since $R_0 \leqslant \langle Z_2^{G_0} \rangle$ this implies $V_0 = [N_0, R_0] \leqslant Z_0$. Hence R_0/V_0 is a covering group of $\mathrm{Sp}(2n, q)'$ and thus the proof of (6.3) shows $R_0/V_0 \simeq \mathrm{Sp}(2n, q)'$. Now $N_0 = V_0$ and (6.4)(1) holds.

Assume $q \geqslant 8$. If A is a complement to $Z_0 \cap Z_2$ in Z_2, then $|A| = q$ and A acts as a transvection group on $N_0/Z(R_0)$. Thus $A \leqslant R_0 C_0$ and (2.8) shows that the extension of R_0/N_0 by $N_0/Z(R_0)$ splits. As $\mathrm{Sp}(2n, q)$ has a trivial multiplier for $q \geqslant 8$ this proves (2).

Now Theorems 1 and 2 are consequences of (3.5), (3.7), and (6.1)–(6.4).

7. Proof of Theorem 3

Assume in this section that the hypothesis of Theorem 3 holds. Pick a $2'$-component K_0 of M_0 satisfying condition $(*)$ of Theorem 3. Then by (3.2), (3.4) $N_1 \leqslant N(K_0)$ and if we set $G_0 = K_0 N_1$, $G_1 = M_1$, $Q_0 = O_2(G_0)$, and $Q_1 = N_1$ then (3.5) shows that the pair G_0, G_1 satisfies Hypothesis C in G. Hence all the results of Sections 4–6 hold. We use the notation introduced in these sections. Especially $d = 2$ or (5.4) holds for G_0 and $d = 4$. If $d = 2$ fix some $\sigma \in G_1$ with $\delta_0^\sigma = \delta_2$. We first show:

(7.1) *If* $\Phi(Q_0 \cap K_0) = 1$, *then* $\Phi(Q_0) = 1$. *Moreover,* $Q_0 = Z_0$ *if* $\bar{G}_0 \simeq \mathrm{SL}_n(2^m)$.

Proof. Assume $N = \Phi(Q_0) \neq 1$ and let $L = N_{G_1}(N)$. Since $N_{M_0}(G_0) = N_{M_0}(K_0 N_1) \leqslant L$, condition $(*)$ of Theorem 3 implies $L \leqslant M_0$.

Suppose first $\bar{G}_0 \simeq \mathrm{SL}_n(2^m)$. Then $d=2$ as $\Phi(Q_0 \cap K_0)=1$ and Z'_0 is irreducible by (4.3) if $n>2$. Thus $[Z'_0, Z'_2]=[Z'_0, Q_1]=[Z'_2, Q_1]$. Claim

$$(*) \quad Q_0 = Z'_0 * C^0,\ C^0 = C_{Q_0}(R_0),\ R_0 = \langle Z_2'^{G_0} \rangle.$$

If $n=2$, then $|Q_0 : C_{Q_0}(Z'_2)| = 2^m$. Since R_0 is generated by two conjugates of Z'_2 our claim follows. If $n=3$ and $m=1$, R_0 is generated by three conjugates of Z'_2, since $L_3(2)$ is generated by three involutions. Hence again our claim follows.

Finally if $n \geqslant 3$ and $m>1$ if $n=3$ lt $\tilde{Q}_0 = Q_0/\Phi(Q_0)$, $\tilde{C}^0 = C_{\tilde{Q}_0}(O^2(R_0))$, and C^0 coimage of $\tilde{C}^0$. Then $\tilde{Q}_0 = \tilde{Z}_0 \times \tilde{C}^0$ and as $C^0 \cap Z'_0 \leqslant \Phi(Q_0) \cap Z'_0 = 1$ we obtain $[C^0, R_0]=1$.

Now $(*)$ shows $\Phi(Q_0)=\Phi(C^0)=\Phi(Q_2)=1$. Hence $C^0 = C_0$, $Q_0 = Z_0$, and (7.1) holds in this case.

Next suppose $\bar{G}_0 \simeq G_2(q)$. Then (6.2) implies $Q_0 Q_2 = (Q_0 \cap Q_2) Z_0 Z_2$. Hence again $\Phi(Q_0)=\Phi(Q_2)$ and $\sigma \in M_0$.

Finally, if $\bar{G}_0 \simeq \mathrm{Sp}(2n, q)$, $q=2$ or 4, then (6.3) implies $Q_i = (Q_i \cap Q_j) Z_i$ for $\{i, j\} = \{0, 2\}$. Hence again $\Phi(Q_0)=\Phi(Q_2)$ and $\sigma \in M_0$, a contradiction.

(7.2) *If* $d=4$ *then* $Q_1 = (Q_1 \cap K_0) * C_0$, $C_0 = C_{Z_0}(K_0)$.

Proof. Pick again $\sigma \in G_1$ with $\delta_0^\sigma = \delta_2$, where $(\delta_0, \delta_1, \delta_2, \delta_3, \delta_4)$ is an arc with $Z_4 \nleqslant Q_0$. By the proof of (5.4) $Q_0 = (Q_0 \cap K_0) * C^0$, $C^0 = C_{Q_0}(K_0)$.

Suppose first $Z'_2 \leqslant Z(Q_0 \cap K_0) C^0$. Then as $[Z'_2, Q_0] = [Z'_2, Q_1] \simeq \mathbb{Z}_2$, $Z'_2 \leqslant C^0$ or $Z'_2 \leqslant Z(Q_0 \cap K_0) C_0 \leqslant Z_0$. In the second case $Q_0 = Q_2$, a contradiction to $Z_4 \leqslant Q_2$. In the first case $Q_0 \cap K_0 \leqslant Q_2$, whence

$$\mathbb{Z}_2 \simeq [Z'_4, Q_2] = [Z'_4, Q_0 \cap K_0] = [Z'_4, Z'_0] \leqslant Z'_0,$$

a contradiction since $[K_0, Q_0 \cap K_0] \nleqslant Z'_0$. So $Z'_2 \nleqslant Z(Q_0 \cap K_0) C^0$. Hence $[Q_1, Z'_2] = [Q_0 \cap K_0, Z'_2] = \Phi(Q_0 \cap K_0)$. This implies $Z'_2 \leqslant (Q_0 \cap K_0) C_0$ and $Z'_2 \cap Z_0 = \Phi(Q_0 \cap K_0)$, since otherwise $\Phi(Q_0 \cap K_0) = [Z_0 \cap Z'_2, Q_1] \leqslant [Q_1, Z_0] = [Q_1, Z'_0]$ a contradiction to the structure of K_0 as described in Theorem 2. We obtain $Q_0 \cap Q_2 = Z'_0 Z'_2 C^0 = Z'_0 Z'_2 C^2$, $C^2 = C_{Q_2}(O^2(G_2))$. This implies $\Phi(C^0) = \Phi(Q_0 \cap Q_2) = \Phi(C^2)$. Now condition $(*)$ of Theorem 3 implies $\Phi(C^0)=1$ as in the proof of (7.1). Thus $C^0 = C_0$.

(7.3) *In case* $\bar{G}_0 \simeq \mathrm{Sp}(6, q)$ *and* $\Phi(Q_0 \cap K_0) \neq 1$ *also* $Q_1 = (Q_1 \cap K_0) * C_0$.

Proof. By Theorem 2, $|Q_0 \cap K_0 : C_{Q_0 \cap K_0}(Z_2)| = q^5$. Hence $Q_0 = (Q_0 \cap K_0)\, C_{Q_0}(Z_2)$. This implies easily $Q_0 = (Q_0 \cap K_0) * C^0$ and $[Z_2, C^0] = 1$. Since $|Z_2 : Z_2 \cap Q_0| = q = |Z_0 : Z_0 \cap Q_2|$ we obtain

$Q_0 \cap Q_2 = C^0 * (Z_2 \cap Q_0)(Z_0 \cap Q_2)$ and thus $\Phi(C^0) = \Phi(C^2) = 1$ by condition $(*)$ of Theorem 3. Hence $C^0 = C_0$.

(7.4) *The following hold*:

(1) $Q_0 = N_0 = Z_0(Q_0 \cap K_0) = C_0(Q_0 \cap K_0)$ *or* $\Phi(Q_0) = 1$, $|Q_0 : (Q_0 \cap K_0)C_0| = 2$, *and* $\bar{G}_0 \simeq \mathrm{Sp}(2n, 2)$.

(2) $G_0/N_0 = E(M_0/N_0)$ *or* $|G_0/N_0 : E(M_0/N_0)| = 2$ *and* $\bar{G}_0 \simeq \mathrm{Sp}(4, 2)$ *or* $G_2(2)$.

(3) $F(M_0/N_0) = F_0/N_0 \leqslant (M_0 \cap M_1)/N_0$.

(4) $M_0 = K_0(M_0 \cap M_1)$.

Proof. By (7.1)–(7.3) either $\Phi(Q_0) = 1$ or $Q_0 = (Q_0 \cap K_0)C_0$. Since $F^*(M_0) = N_0 \leqslant Q_0$ and $Q_0 \cap K_0 \leqslant N_0$, this implies $Q_0 = N_0$. By (7.1)–(7.3) it remains to show $Q_0 = Z_0$ in case $\Phi(Q_0) = 1$ to prove (1).

In case $\bar{G}_0 \simeq \mathrm{SL}_n(q)$ this has been shown in (7.1). Next let $\bar{G}_0 \simeq G_2(q)$. Then $|Q_2 : Q_2 \cap Q_0| = q^3$. Since G_0/Q_0 can be generated by two conjugates of Q_2, we obtain $Q_0 = (Q_0 \cap K_0)C_0$ and (1) holds. If $\bar{G}_0 \simeq \mathrm{Sp}(2n, q)$, $q \geqslant 4$, then as shown in the proof of (6.3), (6.4) $|Q_0 : Q_0 \cap Q_2| = q$. Since G_0 is generated by $2n$-conjugates of Q_2 this again implies $Q_0 = (Q_0 \cap K_0) \cdot C_0$. Since in case $\bar{G}_0 \simeq \mathrm{Sp}(2n, 2)$ we have $H^1(\bar{G}_0, V) \simeq \mathbb{Z}_2$, V the natural $\bar{G}_0$-module, this proves (1).

To prove (2) let R_0 be a $2'$-component of M_0 different from K_0. Then easily $[K_0, R_0] = 1$. Claim $R_0 \leqq C(Z_1)$. If $Q_1 \leqq K_0 Q_0$ then $R_0 \leqq M_0 \cap M_1$ and the claim holds by the hypothesis of Theorem 3. So we may assume $\bar{G}_0 \simeq \Sigma_6$ or $G_2(2)$. But then, as $[K_0 \cap O_0, R_0] = 1$, (1) implies $[Q_0, R_0] \leqq C_0 \leqq Z_1$, whence $R_0 \leqq N_{M_0}(Z_1) \leqq M_0 \cap M_1$ and the claim also holds. Now, since by (1) in any case $|N_0 : (Q_0 \cap K_0)Z_1| \leqq 2$, we obtain $[R_0, N_0] = 1$, a contradiction to $N_0 = F^*(M_0)$.

Statements (3) and (4) are now obvious since $Q_1 = N_0(Q_1 \cap K_0)$ and $M_0 = K_0 N_{M_0}(Q_1 \cap K_0)$ by the Frattini Argument.

(7.5) *If* $Q_0' = 1$, *then* $\bar{G}_0 \simeq \mathrm{SL}_n(q)$.

Proof. Suppose false. Then by Theorem 2 $\bar{G}_0 \simeq G_2(q)$ or $\mathrm{Sp}(2n, q)$. Now by the structure of K_0 and (7.4) in the first case $\mathfrak{A}(Q_1) = Q_0 \cup Q_2$ and in the second case all involutions of $\langle Q_0^{G_1} \rangle$ are contained in $Q_0 \cup Q_2$, since $\langle Q_0^{G_1} \rangle = Q_0 Q_2$, since each Q_0^g, $g \in G_1 - G_0$ induces $\mathrm{GF}(q)$-transvections on Z_0'. Hence in any case $G_1 = (G_1 \cap M_0)\langle \sigma \rangle$, $\sigma^2 \in M_0$, since by condition $(*)$ $M_0 \geqslant N_{G_1}(N_0)$. Since $G_1 \cap G_2 = (G_1 \cap G_0)^\sigma$ acts non-trivially on $Q_0 \cap K_0$ we obtain in any case $G_0 \cap G_1 = G_1 \cap G_2$. Further, as $Q_1 = O_2(G_1 \cap M_0)$, $q = 2$ resp. $q \leqslant 4$ by Theorem 2.

First assume $\bar{G}_0 \simeq G_2(2)$. Then $M_0 = G_0 \cdot L_0$, $L_0 = C_{M_0}(G_0/Q_0)$. Hence

$M_0 \cap M_1 = (G_0 \cap M_1)L_0$ and $Q_1 L_0 = C_{M_0 \cap M_1}(G_0 \cap G_1/Q_1)$. Since σ normalizes $G_0 \cap G_1$ we obtain $O^2(L_0) \lhd \langle M_0, M_1 \rangle = G$. Thus $M_0 = G_0$ and $M_0 \cap M_1/Q_1 \simeq \Sigma_3$. But this is a contradiction to $O_2(M_1) = Q_1$.

Now the same argument shows $G_0 = M_0$ in case $\bar{G}_0 \simeq \mathrm{Sp}(2n, 2)$. Hence in this case $n = 3$ and σ induces an outer automorphism on $G_0 \cap G_1/Q_1$. But since $(G_0 \cap G_1)\langle \sigma \rangle$ acts on Q_1/Q_0Q_2, which is a natural $\mathrm{Sp}(4, 2)$-module for $G_0 \cap G_1$, this is impossible. So finally $G_0 \simeq \mathrm{Sp}(2n, 4)$, $n \geqslant 2$. If $F_0/N_0 \neq 1$, then $1 \neq [C_0, F_0] = [Z_1, F_0] = [Q_1, F_0]$ by (7.4), since by (6.3)(3) G_0/C_0 has no complement to Q_0/C_0. But as $Q_1 F_0 \lhd M_1$, this is impossible. This implies $|M_0 : G_0| \leqslant 2$ and M_0 induces field automorphisms on G_0. If now $M_0 \neq G_0$, then $M_1/Q_1 = (M_0 \cap M_1)/Q_1 \cdot C_{M_1/Q_1}(M_0 \cap M_1/Q_1)$ as shown above. This contradicts $Q_1 = O_2(M_1)$. Hence $M_0 = G_0$ and σ induces a field automorphism on $O^{2'}(G_0 \cap G_1/Q_1) \simeq \mathrm{Sp}(2n-2, 4)$, since $G_0 \cap G_1/Q_1 \simeq \mathrm{Sp}(2n-2, 4) \times \mathbb{Z}_3$ and so σ cannot centralize $O^{2'}(G_0 \cap G_1/Q_1)$ as $Q_1 = F^*(G_1)$. But this is impossible by (2.9).

(7.6) *If $Q_0' = 1$, then Theorem* 3 *holds.*

Proof. By (7.5) $\bar{G}_0 \simeq \mathrm{SL}_n(q)$, $n \geqslant 2$ and $q = 2^m$. If $n = 2$, then $\mathfrak{A}(Q_1) = Q_0 \cup Q_2$. Hence $G_1 = (G_1 \cap M_0)\langle \sigma \rangle$, $\sigma^2 \in M_0$, by condition $(*)$. It is now an easy exercise to show that Theorem 3 holds in this case.

So assume $n \geqslant 3$. Then by (4.3) $Q_0 = Z_0 = Z_0' \times C_0$. As $Q_1 \leqslant K_0 Q_0$ we obtain $Q_1 = (Q_1 \cap K_0) \times C_0$. Now K_0 is the extension of $\mathrm{SL}_n(q)$, $n \geqslant 3$, by its natural module Z_0' and $Q_1 \cap K_0$ is the normal 2-subgroup of the stabilizer of a point Z_1' of Z_0' in this extension.

Hence

$$\Phi(Q_1 \cap K_0) = Z(Q_1 \cap K_0) = Z_1'$$

and either $q = 2$ and $Q_1 \cap K_0$ is extraspecial or the extension of K_0/Z_0' by Z_0' splits by [4]. In the second case K_0 is isomorphic to $O^{2'}(P)$, P a maximal parabolic in $\mathrm{SL}_{n+1}(q)$; whence

$$Q_1 \cap K_0 \simeq \left\{ \begin{bmatrix} 1 & & & \\ a_1 & 1 & & 0 \\ & 0 & \ddots & \\ a_{n-1} & & & \ddots \\ c & b_{n-1} & b_1 & 1 \end{bmatrix} \middle| \; a_i, b_i, c \in \mathrm{GF}(q) \right\}.$$

Hence in any case $Q_1/Z_1 \simeq Q_1 \cap K_0/Z_1'$ carries the structure of a $2n-2$-dimensional non-degenerate orthogonal space of $+$ type over $\mathrm{GF}(q)$, with quadratic form given by the square map. (In case $q > 2$ this is an easy exercise in linear algebra!)

Let $L_1 = C_{M_1}(Q_1/Z_1)$. Then M_1/L_1 is a subgroup of $\Gamma O^+(2n-2, q)$. Now by condition $(*)$ $L_1 \leqslant M_0 \cap M_1$. Suppose $O^2(L_1) \neq 1$. Then $[Z_1, O^2(L_1)] \neq 1$ by Hypothesis (b). Since $O^2(L_1)$ centralizes $Z'_1 = Q'_1$ we obtain

$$1 \neq [Z_1, O^2(L_1)] = [C_0, O^2(L_1)] \lhd G = \langle M_0, M_1 \rangle, \qquad \text{a contradiction.}$$

Hence $L_1 = Q_1$ and $G_0 \cap G_1/Q_1 \lesssim GO^+(2n-2, q)$. Hence by (2.10) one of the following holds:

(a) $O^{2'}(G_0 \cap G_1) \lhd G_1$,

(b) $q=2$, $3 \leqslant n \leqslant 5$, and there exists no $G_0 \cap G_1$-invariant complement to Q_0/Z_1 in Q_1/Z_1. Further $\langle (G_0 \cap G_1)^{G_1} \rangle / Q_1$ is isomorphic to $(\mathbb{Z}_3 \times \mathbb{Z}_3)\mathbb{Z}_2$, A_7 resp. A_9.

Now in case (b) $C_0 = 1$, since $\langle (G_0 \cap G_1)^{G_1} \rangle$ centralizes C_0. Hence $G_0 = M_0$ and case (1) of Theorem 3 holds.

Next if $n \geqslant 4$ in (a), then Q_0/Z_1 and Q_2/Z_1 are the only $O^{2'}(G_0 \cap G_1)$-invariant proper subgroups of Q_1/Z_1. (Since they are dual modules for $O^{2'}(G_0 \cap G_1)/Q_1 \simeq \mathrm{SL}_{n-1}(q)$!) Hence $|G_1 : (M_0 \cap G_1)| = 2$. Let $R_1 = \langle (G_0 \cap G_1)^{G_1} \rangle$. Then $|R_1/O^{2'}(G_0 \cap G_1)|$ is odd and $[Z_1, R_1] = Z'_1 = Z_1 \cap K_0$ if $q > 2$. But then $C_0 = C_{Z_1}(R_1) = C_{Z_1}(G_0 \cap G_1) = 1$, whence $(M_0/Q_0)'$ is an extension of G_0/Q_0 by diagonal and field automorphisms and Theorem 3 holds.

So $q=2$ if $n \geqslant 4$. But then easily $G_0 = M_0$ and $|C_0| \leqslant 2$, since otherwise $C_{C_0}(\sigma) \neq 1$ for $\sigma \in G_1$ with $G_1 = (G_0 \cap G_1)\langle \sigma \rangle$. Hence again Theorem 3 holds.

So we finally have $n=3$. If $q=2$, then $G_0 = M_0$ since $Q_1 = L_1$. Hence $(G_0 \cap G_1) \lhd G_1$ and $G_0 \cap G_1/Q_1 \simeq \Sigma_3$. Since $G_1/Q_1 \lesssim O^+(4,2)$ we obtain $G_1/Q_1 \simeq \Sigma_3 \times \mathbb{Z}_3$. It is now easy to see that either $C_0 = 1$ or $|C_0| = 4$ and so one of the cases of Theorem 3 holds. (Otherwise there must be a four-group normal in G_0 and G_1!)

Hence $q \geqslant 4$. Now $G_1 \cap M_0 = N_{G_1}(Q_0)$ and $G_0 \cap G_1/Q_1 \simeq L_2(q) \times \mathbb{Z}_{q-1} \lesssim \Omega^+(4,q) \times \mathbb{Z}_{q-1} \simeq L_2(q) \times L_2(q) \times \mathbb{Z}_{q-1}$ and the central subgroup $H/Q_1 \simeq \mathbb{Z}_{q-1}$ of $G_0 \cap G_1/Q_1$ does not act by scalar multiplication on Q_1/Z_1. Since $N_{M_1}(Q_0) = M_0 \cap M_1$ and $O_2(M_0 \cap M_1/Q_1) = 1$ this implies that $|R_1/O^{2'}(G_0 \cap G_1)|$ is odd (R_1 as above) and $|M_1 : M_0 \cap M_1| = 2$. Now as above $C_0 = 1$ and so $(M_0/Q_0)'$ is an extension of G_0/Q_0 by diagonal and field automorphisms. Hence again Theorem 3 holds.

(7.7) *Case* (a4) *of Theorem* 1 *does not hold for* $R_0 = O^2(G_0)$.

Proof. Suppose false. Then by (7.4) $C'_0 = C_0 \cap Q'_1 \simeq \mathbb{Z}_2$ and $Z'_1 = Z_1 \cap Q'_1 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $C'_0 = C_0 \cap K_0 \lhd M_0$, this implies $Z'_1 \leqslant Z(M_0 \cap M_1)$.

Hence $M_0 \cap M_1 \lhd M_1$, since $Z_1' \lhd M_1$. Suppose $F_0 \neq Q_0$ and let $F_1/Q_1 = F(M_1/Q_1)$. If $F_0 Q_1 = F_1$, then $1 \neq O^2(F_0) = O^2(F_1) \lhd G = \langle M_0, M_1 \rangle$, a contradiction. Hence $L_1 = N_{F_1}(F_0 Q_1) > F_0 Q_1$. Since L_1 normalizes $O^2(F_0) = O^2(F_0 Q_1)$, condition (*) of Theorem 3 implies $L_1 \lhd M_0 \cap M_1$. But then $L_1 \leqslant F_0 Q_1$ by the structure of $\mathrm{Aut}(G_0/Q_0) \simeq \Sigma_8$, a contradiction.

This implies $F_0 = Q_0$ and so by (7.4) $M_0 = G_0$, since Σ_8 cannot act on $Z_0 \cap K_0$. Let $R = M_0 \cap M_1$ and $\bar{M}_1 = M_1/Q_1$. Then $\bar{R} \simeq L_3(2)$ and by (7.4) and the structure of K_0 visibly Z_0/Z_1 and $Z_2(Q_1)/Z_1$ are equivalent natural $\bar{R}$-modules. Hence no element of $M_1 - R$ can induce an outer automorphism on $\bar{R}$. This implies $\bar{M}_1 = \bar{R} \times C_{\bar{M}_1}(\bar{R})$. Let $\bar{C} = C_{\bar{M}_1}(\bar{R})$ and suppose there exists an element $1 \neq h \in C$ of odd order. Since $M_1 \cap K_0 = O^2(M_1 \cap M_0)$, $Q = Q_1 \cap K_0 \lhd M_1$. As $h \notin C(C_0')$, $o(h) = 3$. Further h does not normalize $Z_0' = Z_0 \cap K_0$ and $Q_0' = Q_0 \cap K_0$. Hence the structure of $M_1 \cap K_0$ implies that h must act fixed-point-freely on Q, a contradiction since $\mathrm{cl}(Q) > 2$.

Thus C is a 2-group, which contradicts $Q_1 = O_2(M_1)$.

(7.8) *If* $\bar{G}_0 \simeq \mathrm{Sp}(6, q)$ *then case* (3) *of Theorem* 3 *holds.*

Proof. Suppose $\bar{G}_0 \simeq \mathrm{Sp}(6, q)$. Then by (7.5) and (7.1) $\Phi(Q_0 \cap K_0) \neq 1$. Hence $Z_1' = Z_1 \cap Q_1' = Z_1 \cap K_0$ by (7.4) and the structure of K_0 described in Theorem 2. Let h be an element of odd order in F_0. Then as $[K_0, h] \leqslant C_0 \cap K_0$ and $C_0' = C_0 \cap K_0 \leqslant K_0'$, $[Z_1', h] = 1$. This implies $F_0 \leqslant C(Z_1')$. Now $C_{M_1}(Z_1') \lhd M_1$ and $C_{M_1}(Z_1') \leqslant M_0 \cap M_1$. Hence the same argument as in (7.7) implies $F_0 Q_1/Q_1 = F((C_{M_1}(Z_1')/Q_1))$ and so $F_0 = Q_0$. This shows that M_0/Q_0 is an extension of $\mathrm{Sp}(6, q)$ by field automorphisms.

Let $\sigma \in M_1 - M_0$. Then Z_0^σ is $M_0 \cap M_1$-invariant. Let $\tilde{Q}_1 = Q_1/Z_1$. Then the action of $G_0 \cap M_1$ on $\tilde{Q}_1$ implies $Z_0^\sigma \nleqslant Q_0$. Now by (5.2) $|Z_0^\sigma : Z_0^\sigma \cap Q_0| = q$ and Z_0^σ induces a root group of transvections on Z_0. As $[Z_0^\sigma, Q_1] = [Z_0^\sigma, Q_0] \trianglelefteq G_0 \cap M_1$, the action of $G_0 \cap G_1$ on $\tilde{Q}_0$ implies that $[\tilde{Z}_0^\sigma, Q_0] = [Z_0^\sigma \cap Q_0]$ is non-equivalent to $[\tilde{Z}_0, Q_1]$ as $G_0 \cap M_1/Q_1 \simeq \mathrm{Sp}(4, q)$. This shows that σ induces an outer diagram automorphism on $G_0 \cap G_1/Q_1$ and thus $M_1 = (M_0 \cap M_1)\langle \sigma \rangle$, since $G_0 \cap G_1 = C_{M_1}(Z_1') \lhd M_1$.

So to prove that case (3) of Theorem 3 holds, it remains to show that $|C_0| = q$. But this follows from $|Z_1 : C_0| = q$ and $C_0 \cap C_0^\sigma \lhd \langle M_0, M_1 \rangle = G$.

Now Theorem 3 is a consequence of (7.4)–(7.8).

8. Proposition 4

(8.1) *Suppose G is a group satisfying* (a)–(e) *of Theorem* 3 *and:*

(+) *There exists no* $A_4 \simeq L_0 \lhd\lhd M_0$ *with* $L_0 \nleqslant M_1$.

Let $\{R_i \mid i \in I\}$, $I = \{1, ..., n\}$ *be the set of* 2'*-components of* M_0, *which are not contained in* M_1. *Then the following hold:*

(1) $I \neq \varnothing$ *and each* R_i *satisfies one of the cases of Theorem* 1.

(2) $[R_i, R_j] = 1$ *for* $i \neq j$.

(3) $R_I = \prod_{i \in I} R_i \trianglelefteq M_0$ *and* $M_0 = R_I(M_0 \cap M_1)$. *Further* $N_1 = (N_1 \cap R_I) N_0$.

Proof. Use the notation of Theorem 3 and let $F_0/N_0 = F(M_0/N_0)$. Then condition (+) and [2, (8.4)] imply $F_0 \leqslant M_1$, since $C(F_0N_1, N_1) \leqslant M_1$. Hence the hypothesis of Theorem 2 holds for $M = M_0 \cap M_1$ by (3.2), (3.3) and, since case (1) of Theorem 2 is not satisfied, $I \neq \varnothing$ and each R_i satisfies one of the cases of Theorem 1. Further by (3.3) $N_1 \leqslant N(R_i)$.

This proves (1). Since $R_i \trianglelefteq E_0$ for each $i \in I$, $E_0/N_0 = E(M_0/N_0)$, (2) is now an easy consequence of the structure of the R_i described in Theorem 1.

As $E_0 = R_I(E_0 \cap M_1)$ and $R_i \cap M_1$ is a maximal parabolic of R_i, there exists a $T \in \mathrm{Syl}_2(E_0)$ with $T \leqslant M_1$. By the Frattini argument $M_0 = E_0 N_{M_0}(T)$. Let $F/N_0 = F^*(M_0/N_0)$. Then, as $[N_1, F_0] \leqslant N_0$, $R_I \trianglelefteq F$, and $N_1 = (N_1 \cap R_I) C_{N_1}(R_I/Q_2(R_I))$, we obtain $N_1 = (N_1 \cap R_I) N_0$. Hence we may assume $N_1 \leqslant T$. But then by condition (e) of Theorem 3, $Z_1 = \Omega_1(Z(T))$ and thus $N_{M_0}(T) \leqslant N_{M_0}(Z_1) \leqslant M_0 \cap M_1$. Since $M_0 \cap M_1 \leqslant N(R_I)$ this proves (3).

If now condition (2) of Proposition 4 holds, then $I = \{1\}$ and by Theorem 2 and (8.1)(3) $M_0 \cap M_1$ is a maximal subgroup of M_0. Especially $R_1 Q_1 \trianglelefteq M_0$ and so condition (*) of Theorem 3 holds.

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